Preliminary Assignment for the research course "Typical Kakeya-type sets"

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First read carefully the research description to understand the background and some of the basic definitions and concepts. If anything in the description or in the assignment is unclear, please feel free to ask me.

Try to solve as many of the following problems as you can. For each problem, briefly indicate whether you solved it completely, solved it partially, have only an idea, or got stuck.

Among the problems for which you have substantial progress, write up detailed solutions of at most three problems, preferably the ones you found the most challenging or interesting. Please send your solutions to me by email.

Please work independently. You may look up definitions and standard background material, but cite any source you use. You are not allowed to use AI tools, such as ChatGPT, Claude, Gemini, etc., to solve, check, or write up the problems.

Your submitted solutions should reflect your own work and your own mathematical thinking. In the first meetings, we will use some of your solutions and ideas as starting points for discussion, with the goal that everyone understands the solutions and learns from the others' ideas. Please be prepared to explain any problem that you indicate as solved, whether or not you wrote up its solution in detail.

1. Look up a definition of Hausdorff dimension online, state explicitly which definition you are using, and prove directly from this definition that if X and Y are metric spaces and X can be mapped onto Y by a Lipschitz map then $\dim_{\mathrm{H}} Y \leq \dim_{\mathrm{H}} X$, where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.
2. In a topological space a set is called G_δ if it can be obtained as a countable intersection of open sets. Prove that in a complete metric space X a set $E \subset X$ is residual if and only if it contains a dense G_δ set of X.

3. Let $\mathcal{K}_n$ be the set of nonempty compact subsets of $\mathbb{R}^n$, and consider the Hausdorff metric d_H on $\mathcal{K}_n$.

   (a) Prove that for any s and m the sets

   $$\{K \in \mathcal{K}_n \colon \mathcal{L}(K) \leq m\} \qquad \text{and} \qquad \{K \in \mathcal{K}_n \colon \dim_{\mathrm{H}} K \leq s\}$$

   are G_δ subsets of $\mathcal{K}_n$, where $\mathcal{L}$ denotes the Lebesgue measure.

   (b) Let X be a compact subset of $\mathcal{K}_n$ and suppose that for any $K \in X$ and $\delta > 0$ we can find $K' \in X$ such that $d_H(K, K') < \delta$ and $\mathcal{L}(K') \leq m$. Prove that this implies that a typical $K \in X$ has Lebesgue measure at most m.

4. Let

   $$X = \{K \subset [0,1] \times [0,1] \colon K \text{ compact}, \operatorname{proj}_1 K = [0,1]\},$$

   where proj_1 denotes the projection to the first coordinate. Consider the Hausdorff metric on X. Find the Lebesgue measure and the Hausdorff dimension of a typical $K \in X$.

5. Let C be a compact subset of the plane and let E be the union of all lines of the form $y = ax + b$ such that $(a, b) \in C$. (In other words, E is the union of the lines "coded" by the "code-set" C.) For every angle φ let L_φ be the line through the origin in direction φ and let $\operatorname{proj}_\varphi C$ be the orthogonal projection of C to L_φ.
   Prove that E has (2-dimensional) Lebesgue measure zero if and only if the (1-dimensional) Lebesgue measure of $\operatorname{proj}_\varphi C$ is zero for almost every φ.
   Hint: Study the intersection of E with vertical lines and use Fubini's theorem.

Have fun!