VARIATIONS OF THE PLANK PROBLEM

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1. Description of the problem

Plank problems form a classical topic in convex geometry. A *plank* (or strip) in the plane is the region between two parallel lines, whose distance apart is called the *width* of the plank. The original plank problem of Tarski, which dates back roughly a century, reads as follows.

Theorem 1. *If a circular disc D is covered by a finite collection of planks, then the sum of the widths of these planks is at least the diameter of D.*

Although this question has a very elegant solution, its generalization for arbitrary planar convex sets is not so easy to prove, and that was achieved only by Bang in the 1950's. (A set is *convex* if together with any two of its points, it also contains the line segment connecting these points). Bang extended the result for arbitrary convex bodies in $\mathbb{R}^d$, and at the same time found a natural generalization of the problem which, in full generality, has been open ever since. His plank covering theorem is this:

Theorem 2. *If a convex body K is covered by a finite collection of planks, then the sum of the widths of these planks is at least the minimal width $w(K)$ of K.*

Above, a convex body K is a compact, convex set with nonempty interior, and its *width* $w(K)$ is the width of the smallest plank that covers it. In higher dimensions, planks are analogously defined as the region between two parallel (hyper)planes.

In the research project, we will concentrate on a variant of the question suggested by A. Bezdek. What happens if one is not required to cover the entire disc, but only an outer annulus of it? Let D now denote the unit disc centered at the origin, i.e. the disc of radius 1 with center $(0, 0)$.

Research Question 1. *Let $0 < \varepsilon < 1$ be a given parameter. Is it true that if ε is sufficiently small, then if a finite collection of planks covers the annulus $D \setminus \varepsilon D$, then the sum of widths of the planks is still at least 2?*

Bezdek proved the statement for a square instead of the disc. However, as conjectured, his result is not the strongest possible. Our second goal is thus to prove the conjectured optimal bound.

Research Question 2. *Prove that if from a square Q of side length 1, another homothetic square Q' of side length at most $\frac{1}{2}$ is cut, then covering the remaining square annulus by planks still requires total width at least 1.*

That Q' is homothetic to Q means that the two squares are aligned, i.e. their respective sides are parallel to each other. Bezdek proved the above result with the constant $\frac{1}{\sqrt{2}}$ instead of $\frac{1}{2}$.

The most general question can be formulated as follows.

Research Question 3. *Which convex sets C have the property that there exists a bound $\delta(C) > 0$ so that for any positive $0 < \varepsilon < \delta(C)$, no matter how we take a homothetic image εC of C, covering the annulus $C \setminus \varepsilon C$ by planks still requires total width $w(C)$?*

For brevity, let us refer to the above property as the *annulus property.* White and Wisewell characterized planar polygons for which the answer to Question 3 is affirmative, while in a previous BSM REU project we managed to find a large class of convex bodies in 2- and 3-space which do not possess the annulus property. Apart from Research Questions 1 and 2, the main goal of the current project would be to prove positive results, that is, to prove that the annulus property holds for a large class of convex bodies in 2- and 3-space.

2. Prerequisites

Although the problem sounds very elementary, the applied tools are based on various methods from geometry, analysis, and combinatorics. That said, we are going to start from the basics and cover the topics that are needed. Completion of an introductory calculus course is mandatory; a course in elementary geometry is preferred.

3. Qualifying problems

Participation in the research project will be offered based on the solutions for the set of qualifying problems below (note that you can submit partial solutions as well). Please work individually and submit your solutions by August 31, 2025 using the email address `ambruge AT gmail DOT com`.

Qualifying Problem 1. *Find an $\varepsilon < 1$ for which the statement of Question 1 does not hold. Try to make ε as small as possible.*

Qualifying Problem 2. *Prove Theorem 1 by taking a ball in $\mathbb{R}^3$ whose equator is the boundary of D, then projecting the planks onto the boundary of the ball and calculating the amount of surface area covered by such a spherical plank.*

Qualifying Problem 3. *Find a triangle T for which the annulus property (i.e. the property in Research Question 3) does not hold. Can you characterize triangles that possess this property?*

Qualifying Problem 4. *Assume that the planks $P_1, \dots, P_n$ in the plane are all* centered*, that is, the origin lies on their middle lines. Let v_i denote the* cross normal vector *of the plank P_i, i.e. a vector that is perpendicular to the bounding lines of P_i and whose length is the width of P_i. Suppose that a finite point set $\mathcal{X}$ has the property that for any $x \in \mathcal{X}$ and for any $i \in \{1, \dots, n\}$, $x + v_i$ or $x - v_i$ is contained in $\mathcal{X}$ (it is possible that both are so). Prove that there is a point in $\mathcal{X}$ that is not covered by any P_i.* (Hint: How do the norms of x and $x \pm v_i$ compare to each other? Can you jump from a covered point to an uncovered one?)