Artificial Intelligence and Unit Distance Graphs

Let $X \subseteq \mathbf{R}^n$ be a finite subset. We can give X the structure of an undirected graph called a *Unit Distance Graph (UDG)* by letting the edge set comprise those vertex pairs $\{x, y\} \subseteq X$ that are unit distance apart: $|x - y| = 1$. The question of what is the maximum number of edges $u(n)$ of a UDG of a given number of vertices n is still open since was posed by Erdős in 1946 [1]. The number of vertices for which an exact answer is known was recently raised to $n \leq 21$ [2]. Even the asymptotic behavior is unclear as the known upper and lower bound are quite far apart: see [3] resp. [4] for the planar, and [5] resp. [6] for the spatial case.

The aim of this project is to study UDGs that have a high edge count. We try to use computer search. This is an ongoing project, involving several BSM students and multiple members of the AI group at the Rényi Institute.

With the Spring and Summer 2023 groups, we developed a computer search algorithm that could find all the densest known planar UDGs in [3, Table 1], and moreover go on and find dense UDGs up to vertex number 100. We wrote a paper that describes our method [7].

With the groups that came after, we were working on multiple directions to improve the result and extend it to other Euclidean spaces like the 2-sphere or 3-dimensional space. I will be happy to expound in person on the many great ideas that recent groups came up with and explored.

This semester, our primary focus will be algorithmic improvements on the method desribed in [7]: can we get UDGs with more edges if we change the search space or the search algorithm?

Prerequisites

Strong command of the `Python` numerical library `numpy`.

Qualifying problem

Write a function

```
get_unit_length_count_in_grid(
    coeff_abs_max: int,
    generators: np.ndarray
) -> int:
```

that given

1. A maximum coefficient absolute value `coeff_abs_max` $\in \mathbf{Z}_{>0}$ and
2. A complex vector `generators` $= (z_1, \ldots, z_k)$,

outputs the number of entries of unit length, up to numerical precision, in the grid

$$\{c_1 z_1 + \cdots + c_k z_k : c_i \in \{-\texttt{coeff\_abs\_max}, -\texttt{coeff\_abs\_max} + 1, \ldots, \texttt{coeff\_abs\_max}\}, 1 \leq i \leq k\}.$$

To improve your evaluation, you can try to *vectorize* your solution, that is use as few explicit iterations (such as ones using `for`, `while`, `map`, recursion, etc.) along array dimensions as possible. This will greatly improve the speed of your code.

To hand in the qualifying problem, you can write the function in the file `qualifying_problem_spring_2025.py` included in the research project description. The file includes test cases, you just have to run it to test your solution.

You can hand in multiple versions. I will evaluate the latest submission. A valid submission must arrive to my email address by the deadline written in the general RES course description on the BSM webpage. You can expect me to answer a question before this time if it arrived to my email address at least a day before this deadline.

In your email, please also write me the following:

1. Your Mathematics and Computer Science background.
2. Your Mathematics and Computer Science interests.
3. What do you find especially interesting in this project?

Have fun with the problem and hope to see you in the group!

Contact

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References

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[6] Pál Erdős. On sets of distances of n points in euclidean space. *Magyar Tudományos Akadémia Matematikai Kutatóintézet Közleményei*, 5:165–169, 1960. URL http://www.math-inst.hu/~p_erdos/1960-08.pdf.

[7] Peter Engel, Owen Hammond-Lee, Yiheng Su, Dániel Varga, and Pál Zsámboki. Diverse beam search to find densest-known planar unit distance graphs, 2024. URL https://arxiv.org/abs/2406.15317. Under review.