Connectivity of graphs

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BSM Research opportunities project (Fall 2025)

Introduction

The goal of this research project is to study some well-known old conjectures in graph theory and to extend and improve on the existing partial results. These conjectures are about connectivity properties of undirected graphs.

The edge- and the vertex-connectivity of a graph are among the most important graph parameters. We say that a graph $G = (V, E)$ is *k-edge-connected* (resp. *k-vertex-connected*) if $G - F$ (resp. $G - S$) is connected for all $F \subseteq E$ with $|F| \leq k - 1$ (resp. $S \subseteq V$ with $|S| \leq k - 1$). In other words, it is not possible to disconnect G by deleting less than k edges (or vertices).

Menger's fundamental theorem says that $G = (V, E)$ is k-edge-connected (k-vertex connected) if and only if there exist k pairwise edge-disjoint (pairwise internally vertex-disjoint, resp.) paths from u to v, for all pairs $u, v \in V$. There exist efficient algorithms for testing whether a graph is k-edge- or k-vertex connected, for example by using network flows.

Here we mention three conjectures. The first two conjectures, due to L. Lovász and C. Thomassen, respectively, state that if the vertex connectivity of the graph is sufficiently large, then it contains a k-connected subgraph that satisfies certain properties. (The vertex set of a (sub)graph H is denoted by $V(H)$.)

Conjecture 1 *For every positive integer k there exists an integer $f(k)$ such that if G is $f(k)$-vertex connected, then for every pair $u, v \in V(G)$ there is a path P from u to v such that $G - V(P)$ is k-vertex connected.*

Conjecture 2 *For every positive integer k there exists an integer $g(k)$ such that if G is $g(k)$-vertex connected, then G has a bipartite k-vertex connected spanning subgraph.*

The third conjecture, due to W. Mader, is about removable trees in k-vertex connected graphs with high minimum degree.

Conjecture 3 *For any $k \geq 1$ and any tree T on m vertices, every k-connected graph G with minimum degree at least $\lfloor \frac{3k}{2} \rfloor + m - 1$ contains a subtree T' isomorphic to T such that $G - V(T')$ is k-connected.*

There are some partial results in the literature, but the first two conjectures are still open for $k \geq 3$, and the third conjecture is still open for $k \geq 4$.

The algorithmic aspects of these conjectures are also interesting (i.e. how to find the above paths or subgraphs or trees). Finding lower bounds on the functions $f(k)$ and $g(k)$ is a good problem, too.

Methods and prerequisites

It is a graph theory project, so the participants must be familiar with the basics of graph theory. It is useful to study the fundamental definitions and results concerning graph connectivity before the start of the project (for example, Menger's theorem).

If you have time you may try to show that $f(1) = 3$.

Reading the following sections of this book may be a good start (but most of the other graph theory books also contain the basics of graph connectivity): R. Diestel, *Graph Theory*, Springer. See Sections 3.1-3.3.

Qualifying Problems

Solve the next five exercises, and hand in the solutions by email before August 31, 2025.

Exercise 1. Let T be a tree with $2k$ leaves. Prove that T contains k pairwise edge-disjoint paths such that the end-vertices of each path are leaves of T.

Exercise 2. Suppose that in a directed graph $D = (V, A)$ there exist k pairwise internally vertex-disjoint paths from a designated root vertex r to every other vertex $v \in V - r$. Show that D has a spanning subdigraph $D' = (V, A')$ in which there exist k pairwise internally vertex-disjoint paths from r to every other vertex $v \in V - r$ and the number of arcs entering a vertex v is equal to k for each $v \in V - r$.

Exercise 3. Verify Conjecture 2 for $k = 2$ by showing that $g(2) = 3$ works (and that 3 is the smallest integer that satisfies the requirement).

Exercise 4. Prove that the edge set of a graph G can be partitioned into (the edge sets of) two spanning trees of G if and only if G can be obtained from a single vertex by applying the following operations:
(i) add a new vertex v to the graph and two new edges incident with v,
(ii) delete an edge xy from the graph, and add a new vertex v and three new edges vx, vy, vz incident with v (thus the three new edges must include the edges from v to the endvertices of the deleted edge).

Exercise 5. Prove that Conjecture 3 holds in the special case when $k = 2$ and $m = 1$.

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