Preliminary assignment for the research course “Erdős similarity conjecture and related problems ”

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Solve as many of the following problems as you can and send me your solutions by email until June 1st.

Don't hesitate to ask me if you need clarification or you have any questions. If you send me some of your solutions or partial solutions early then you will get early feedback.

1. Prove that if $E \subset \mathbb{R}$ is a Lebesgue measurable set of positive Lebesgue measure then for every $\varepsilon > 0$ there exists an interval I such that
$$\frac{\lambda(E \cap I)}{\lambda(I)} > 1 - \varepsilon,$$
where λ denotes Lebesgue measure.

2. Prove that if $E \subset \mathbb{R}$ is a Lebesgue measurable set of positive Lebesgue measure then it contains a similar copy of any finite set $F \subset \mathbb{R}$; in other words, there exist $a, b \in \mathbb{R}$, $a \neq 0$ such that $aF + b \subset E$.

3. Construct a Lebesgue measurable set $E \subset \mathbb{R}$ that contains no (non-constant) infinite geometric progression but
$$\limsup_{t \to 0+} \frac{\lambda(E \cap [-t, t])}{2t} = 1.$$
Can we replace lim sup by lim ?

Have fun!