Extremal problems in combinatorial rigidity

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BSM Research opportunities project

Introduction

A d-dimensional *bar-and-joint framework* (or simply *framework*) consists of rigid (fixed length) bars that meet at universal joints. The joints can move continuously in d-space so that the bar lengths and bar-joint incidences must be preserved. The framework is said to be *rigid* (in dimension d) if every such continuous motion results in a congruent framework, that is, the distance between each pair of joints is unchanged. The framework is said to be *globally rigid* if every other framework with the same graph and same bar lengths is congruent with it. Thus global rigidity implies rigidity.

In the *graph* of the framework vertices correspond to the joints and edges correspond to the bars. It is known that if the framework is in sufficiently general (called *generic*) position then (global) rigidity depends only on its graph.

A celebrated result of A. Cauchy from the 19th century states that a triangulated convex polyhedron (that is, the framework formed by its 1-skeleton) in three dimensions is rigid. The 1-skeleton of such a polyhedron is a *triangulation* (or maximal planar graph).

A recent research direction is to consider *braced polyhedra*, in which some additional edges are added to a triangulated convex polyhedron. It has been observed that they have stronger rigidity properties, especially if we assume that the vertex connectivity of their graph is higher. There are some challenging conjectures in this area.

We may also consider these stronger rigidity properties in general, not just in the case of braced polyhedra. This leads us to some extremal questions. For example, what is the minimum number of edges in a graph, in terms of the number n of vertices and k and d, if it remains (globally) rigid in d dimensions even if any subset of at most k edges (or vertices) is deleted?

We shall also be interested in (special cases) of this general extremal problem.

Open problems

Rigidity theory is in the intersection of geometry, algebra, and combinatorics with several applications. It also includes quite a few questions concerning efficient algorithms. We shall chose an open problem suitable for the interested students. Two candidates are given below.

Problem 1 *Let G be a 5-connected braced triangulation with at least two bracing edges. Is it true that $G - \{e, f\}$ is rigid (resp. $G - \{e\}$ is globally rigid) in three-space for every pair e, f of edges (resp. for every edge e) of G?*

Problem 2 *What is the minimum number of edges in a graph G on n vertices for which $G - v$ is globally rigid in d dimensions for all $v \in V(G)$?*

Methods and prerequisites

We shall mostly use graph theoretic and combinatorial methods, so familiarity with the basics of graph theory is useful. Depending on the problems, in some cases geometric intuition is also useful, as well as familiarity with elementary linear algebra.

Basic definitions

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to the d-dimensional Euclidean space R^d. We consider the framework to be a straight line *realization* of G in R^d. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks (G, p) and (G, q) are *equivalent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $uv \in E$, where $dist(x, y)$ denotes the Euclidean distance between points x and y in R^d. Frameworks (G, p), (G, q) are *congruent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of R^d. We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p).

A *motion* (or *flex*) of (G, p) to (G, q) is a collection of continuous functions $M_v : [0, 1] \to R^d$, one for each vertex $v \in V$, that satisfy

$$M_v(0) = p(v) \text{ and } M_v(1) = q(v)$$

for all $v \in V$, and

$$dist(M_u(t), M_v(t)) = dist(p(u), p(v))$$

for all edges uv and for all $t \in [0, 1]$. The framework (G, p) is *rigid* if every motion takes it to a congruent framework (G, q).

Qualifying problems

Solve at least four out of the next five exercises by the deadline. Recall that a graph G is said to be *k-vertex connected* (or simply *k-connected*), for some positive integer k, if G has at least $k + 1$ vertices and $G - X$ is connected for all subsets X of the vertex set of G with size less than k.)

Exercise 1. Show that every triangulation is 3-connected.

Exercise 2. Prove that the smallest 5-connected triangulation is the graph of the icosahedron.

Exercise 3. (a) Find an infinite family of 4-connected triangulations. (b) Find some 5-connected triangulations on more than 12 vertices.

Exercise 4. Describe the set of 6-connected triangulations.

Exercise 5. Can you characterize the rigid frameworks in R^1?

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