Cover graphs and universal rigidity

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BSM Research opportunities project

Introduction

The goal of this research project is to study the potential connections between cover graphs and universally rigid graphs and to obtain new results and examples in at least one of these areas of graph theory and combinatorial rigidity, respectively.

In what follows we present the basic notions and some of the open questions.

Frameworks and universal rigidity

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to the d-dimensional Euclidean space $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks (G, p) and (G, q) are *equivalent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $uv \in E$, where $dist(x, y)$ denotes the Euclidean distance between points x and y in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$. We say that (G, p) is *globally rigid* in $\mathbb{R}^d$ if every d-dimensional framework which is equivalent to (G, p) is congruent to (G, p). We obtain an even stronger property by extending this condition to equivalent realizations in any dimension: we say that (G, p) is *universally rigid* if it is a unique realization of G, up to congruence, with the given edge lengths, in all dimensions $\mathbb{R}^d$, $d \geq 1$.

It is NP-hard to decide if even a 1-dimensional framework is globally rigid. The complexity of the corresponding decision problem for universal rigidity seems to be open, even for $d = 1$. These problems become more tractable, however, if we assume that there are no algebraic dependencies between the coordinates of the points of the framework. A framework (G, p) is said to be generic if the set containing the coordinates of all its points is algebraically independent over the rationals. It is well-known that the global rigidity of frameworks in $\mathbb{R}^d$ is a generic property, that is, the global rigidity of (G, p) depends only on the graph G and not the particular realization p, if (G, p) is generic. This property does not hold for universal rigidity, even if $d = 1$, which follows by considering different generic realizations of a four-cycle on the line. A graph G is called generically globally rigid, (resp. generically universally rigid) in $\mathbb{R}^d$ if every d-dimensional generic framework (G, p) is globally rigid (resp. universally rigid). We shall also use the shorter versions d-GGR and d-GUR, respectively, for these families of graphs. d-GGR graphs are well-characterized for $d \leq 2$. It remains an open problem to extend these results to higher dimensions or to characterize d-GUR graphs for any $d \geq 1$.

Cover graphs

Since it is probably difficult to characterize 1-GUR graphs, special families of 1-GUR (or not 1-GUR) graphs may be of interest. In this context we offer the study of the following family of graphs as a candidate for being not 1-GUR. An *orientation* of a graph G is a directed graph obtained from G by replacing each edge uv by one of the directed edges (arcs) uv or vu. Let $G = (V, E)$ be a graph and let $\bar{G}$ be an acyclic orientation of G. An edge e of G is *dependent* if the reversal of e in $\bar{G}$ creates a directed cycle. An acyclic orientation without dependent edges is called *strongly acyclic*. We say that G is a *cover graph* if G has a strongly acyclic orientation. (It is known that G is a cover graph if and only if it is the Hasse diagram of some partially ordered set on V.) All bipartite graphs are cover graphs: orient all edges from one colour class to the other. Note that cover graphs are triangle-free. It is NP-hard to test whether a given graph is a cover graph. It is known that triangle-free planar graphs (and more generally, triangle-free 3-colorable graphs) are cover graphs. (Recall that by a theorem of Grötzsch, every triangle-free planar graph is 3-colorable.)

Open problems

We shall focus on two or three open problem suitable for the interested students. Some examples are given below.

First we recall a conjectured inductive construction of 1-GUR graphs.

Conjecture 1 *A graph G on at least three vertices is* 1*-GUR if and only if G can be obtained from K_3 by the following operations:*
(i) add an edge,
(ii) choose two graphs G_1, G_2 built by these operations, choose two sets $U_1 \subseteq V(G_1)$, $U_2 \subseteq V(G_2)$ with $|U_1| = |U_2| \geq 2$, delete all edges joining the vertices of U_1 in G_1, then glue the two graphs together along the vertices in U_1 and U_2.

The "if" direction of the conjecture has been verified. Note that the graphs built up from a triangle by operations (i) and (ii) must contain a triangle. Thus finding triangle-free 1-GUR graphs would be interesting.

Minimally 1-GUR graphs, for which the deletion of any edge makes them not 1-GUR, are also interesting. These graphs may be sparse and may have small vertex separations, along which they may be decomposable by the inverse operation of glueing (as in the conjecture above).

Question 2 *Let $G = (V, E)$ be a minimally* 1*-GUR graph. Is there an upper bound on $|E|$ as a linear function of $|V|$?*

Special families which are (not) 1-GUR would also be interesting.

Question 3 *Is it true that no triangle-free planar graph (or even triangle-free* 3*-colorable graph) on at least three vertices is* 1*-GUR?*

We may also ask whether all non-cover graphs are 1-GUR. An interesting graph to analyse is the Grötzsch graph, which is triangle-free and 4-chromatic. It has been shown that this graph is not a cover graph.

Question 4 *Is the Grötzsch graph* 1*- GUR?*

Since this graph is triangle-free, an affirmative answer would disprove the conjecture above

Methods and prerequisites

We shall mostly use graph theoretic and combinatorial methods, so familiarity with the basics of graph theory is useful. Depending on the problems, in some cases geometric intuition is also useful, as well as familiarity with elementary linear algebra.

Qualifying problems

Solve at least three out of the next four exercises to participate in the research project, by the deadline.

Qualifying problem 1. Let G be a connected graph on n vertices and with n edges. Show that G has an orientation $\bar{G}$ in which every vertex v has in-degree exactly one (that is, there is exactly one directed edge uv whose head is v).

Qualifying problem 2. Prove that a graph G has a universally rigid realization (G, p) in $\mathbb{R}^1$ in which $p(u) \neq p(v)$ for all $u, v \in V(G)$ if and only if G is 2-connected. Hint: start with the cycles.

Qualifying problem 3. Suppose that D is an acyclic directed graph with a designated vertex s. Show that D has a spanning arborescence rooted at s if and only if the in-degree of each vertex v, $v \neq s$, is at least one. (An arborescence rooted at s is an oriented tree in which each vertex can be reached from s on a directed path. It is spanning, if it contains all vertices of G.)

Qualifying problem 4. Prove that a graph G has a universally rigid realization (G, p) in $\mathbb{R}^2$ in which $p(u) \neq p(v)$ for all $u, v \in V(G)$ if and only if G is a complete graph.

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