HELLY NUMBERS OF DISCRETE SETS

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Convexity plays a central role in geometry as well as in computer science. In this project, we will focus on combinatorial aspects of convexity, by studying doscretized versions of Helly's theorem. Let us start with the central concepts.

Let C be a point set in $\mathbb{R}^d$. We say that C is *convex*, if along with any two points of it, C also contains the line segment between these two points. Equivalently, C is convex iff it is closed under taking finite *convex combinations*: for any $x_1, \ldots, x_n \in C$, and any set of scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for every i and $\sum_{i=1}^n \lambda_i = 1$, we have that

$$\sum_{i=1}^{n} \lambda_i x_i \in C\,.$$

Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ be convex sets in $\mathbb{R}^d$. Helly's theorem, one of the cornerstones of combinatorial convexity, states the following:

Theorem 1. *If any $d+1$ sets of the family $\mathcal{C}$ share a common point, then there is a point common to all members of $\mathcal{C}$.*

The conclusion may also be written as $\cap\mathcal{C} \neq \emptyset$.

Helly's theorem has been generalized in many directions. There exist versions where convexity of the sets is not required, or where the ambient space differs from $\mathbb{R}^d$.

In the research project, we will be interested in the following problem. Let $\mathcal{S} \subset \mathbb{R}^d$ be a *discrete* set – that is, a set with no accumulation points. A typical example is that of the integer lattice $\mathbb{Z}^d$ consisting of all points with only integer coordinates. Now, we can ask for the *Helly number of* $\mathcal{S}$ which is defined to be the smallest positive integer n for which the restricted version Theorem 1 holds under the restriction that only common points belonging to $\mathcal{S}$ are taken into account:

Definition 1.1. *For a discrete set $\mathcal{S} \subset \mathbb{R}^d$, let $H(\mathcal{S})$ denote the smallest positive integer such that for any family $\mathcal{C}$ of convex sets in $\mathbb{R}^d$ for which the intersection of any $H(\mathcal{S})$ or fewer members contain a point of $\mathcal{S}$ in common, then there exists a point of $\mathcal{S}$ common to all members of $\mathcal{C}$.*

The study of Helly numbers of discrete point sets have been very active recently. One of the important tools is due to Hoffmann:

Proposition 1.1. *If $\mathcal{S} \subset \mathbb{R}^2$ is a discrete set, then $H(\mathcal{S})$ equals the maximum number of vertices of an empty convex polygon in $\mathcal{S}$, that is, a convex polygon with vertices from $\mathcal{S}$ which does not contain any further point from $\mathcal{S}$.*

A higher dimensional version also holds, where one has to seek the maximum number of vertices of an *empty convex polytope* with vertices in $\mathcal{S}$.

The first introductory problem asks for applying this proposition to the case when $\mathcal{S}$ is the planar integer lattice.

Qualifying Problem 1. *Find the Helly number of the integer lattice* $\mathbb{Z}^2$*.*

To give a hint, you may check Pick's theorem in case you are not familiar with it.

The Helly number of $\mathbb{Z}^d$ has been determined for all $d \geq 2$. The next intro problem is to go for the $d = 3$ case.

Qualifying Problem 2. *Prove that* $H(\mathbb{Z}^3) = 8$*.*

The hint here is as follows. On the one hand, find an empty polytope with 8 integer vertices. On the other hand, prove that this is the maximum number of vertices by showing that if the polytope has at least 9 vertices, then the midpoint of the segment connecting some two of these has only integer coordinates – thinking modulo 2 might help here!

Based on the previous two problems, you may also attack the question in all dimensions – note that this is just an extra problem:

Extra Qualifying Problem. *Calculate* $H(\mathbb{Z}^d)$*.*

In the project, we will take $\mathcal{S}$ to be a variety of discrete sets in the plane. We will work on the case when S is a direct product: $S = A \times B$ with $A, B \subset \mathbb{R}$ being discrete sets. A variant of this problem was studied, among others, by a former BSM student in an REU project! We shall first concentrate on the case when $A = \{n^\alpha : n \in \mathbb{N}\}$ and $B = \{n^\beta : n \in \mathbb{N}\}$ with some positive parameters $\alpha, \beta > 0$. Then we can move on to other discrete sets with nice structural properties, e.g. the *stretched grid* for which the set of y-coordinates are increasing exponentially faster than the x-coordinates.

Here is a final introductory problem, for which I do not expect a definite answer, rather just looking for some constructions.

Qualifying Problem 3. *Let* $A = \{2^n : n \in \mathbb{N}\}$*, and set* $\mathcal{S} = A \times A$*. Give a nontrivial lower bound on* $H(\mathcal{S})$ *which is as good as you can get.*

You may also think of giving an *upper bound.*

There are a number of interesting questions in this area, so an enjoyable – and hopefully productive – research project is guaranteed! Of course, we will spend the first half of the project learning all the necessary framework and methods.

Prerequisites: Calculus 1, some knowledge in geometry and combinatorics is preferred – however, we are going to cover all the material needed in the first part of the project.

If you are interested in participating the research project, please send your solutions to the above 3 (+1) Introductory Problems to the email address ambruge@gmail.com.