Preliminary assignment for the online research course “Lebesgue measure and Hausdorff dimension of unions of lines or planes”

BSM, 2021 Summer
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Solve as much as you can and send me your solutions by email. Don't hesitate to ask me if you need clarification or you have any question. If you send me some of your solutions or partial solutions early then you get early feedback.

1. Find the definition(s) of Hausdorff dimension in the internet, and (directly from one of the equivalent definitions) prove that a plane has Hausdorff dimension at most 2.

2. Find an example that shows that in $\mathbb{R}^3$ the union of a 2-parameter family of lines can have Hausdorff dimension less than 3.

3. Let C be a compact subset of the plane and let E be the union of all lines of the form $y = ax + b$ such that $(a, b) \in C$. (In other words, E is the union of the lines “coded” by the “code-set” C.) For every angle φ let L_φ be the line through the origin in direction φ and let $\mathrm{proj}_\varphi C$ be the orthogonal projection of C to L_φ.

   Prove that E has (2-dimensional) Lebesgue measure zero if and only if the (1-dimensional) Lebesgue measure of $\mathrm{proj}_\varphi C$ is zero for almost every φ.

   Hint: Study the intersection of E with vertical lines and use Fubini theorem.

Have fun!