The Turán number of C_{13}

On-line research venture, BSM 2021 SUMMER
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0.1 Requirements

Knowledge of basic Graph Theory is important. I select participants according to the scores on the exercises (subsection 0.4). Send to me by email your solutions not later than May 15, 10 p.m. (according to your time zone). If something is not clear, do not hesitate to ask for clarification.

0.2 3-graphs, 3-trees, Turán number

A 3-*graph* $H = (V, E)$ is a pair, where V is the set of vertices and E (the set of edges) is a set of some unordered **triples** of V such that **any two edges intersect in at most one vertex**. On page 3 you see the catalogue of 3-graphs with 2,3,4 edges. The convention used is that edges are represented by straight line segments.

Some of the 3-graphs on page 3 have more descriptive names, for example A_2, B_3, C_7 are called *stars*, A_2, B_4, C_9 are *paths*, B_5, C_{10} are *cycles.* The 3-graph C_{16} is known as the Pasch configuration. For C_{15} even two names are used: *sail* and (upside down view) *fan*.

The leading role in our research plan is played by C_{13}, it would deserve a more descriptive name - think about it and make a suggestion! (Exrcise 7.)

Two famous 3-graphs, the Fano plane and the affine plane are shown on page 4, they cannot be represented by straight line segments only, some edges have to be represented by curves. (To compensate the shortcomings of my drawings, check that the seven edges of the Fano plane are $\{1, 2, 4\} + i \pmod 7$. Moreover, the three red edges of the affine plane are $\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}$ and the three green edges are $\{3, 5, 7\}, \{2, 4, 9\}, \{1, 6, 8\}$.)

A 3-*tree* is a 3-graph which can be built from the one edge 3-tree by adding at each step a new edge that intersect the previous tree in exactly one vertex.

Let F be a fixed 3-graph. A 3-graph H is called F*-free*, if H has no subgraph isomorphic to F. The *Turán number of* F, $\mathrm{ex}(n, F)$, is the maximum number of edges in an F-free 3-graph on n vertices.

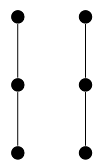
0.3 The aim of the venture

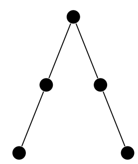
Among the 3-trees with at most four edges there is one, C_{13}, whose Turán number is not known (see Exrcises 5,6). We try to get close to its probable value, $\sim \frac{3n}{2}$.

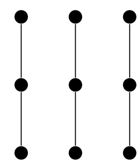
0.4 Exercises

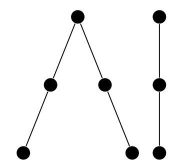
Exercise 1. Prove or disprove: H is a 3-tree if and only if H can be obtained from a graph tree T by extending the edges of T with distinct new vertices. (5 points)
Exercise 2. Prove that $\mathrm{ex}(n, C_7) \leq n$. (5 points)
Exercise 3. Prove that if $\mathrm{ex}(n, C_7) = n$ for some n then $\mathrm{ex}(n+6, C_7) = n+6$. (5 points)
Exercise 4. Prove that $\mathrm{ex}(n, B_4) \leq n$. When do we have equality? (10 points)
Exercise 5. Prove that $\mathrm{ex}(n, C_{13}) \geq 6\lfloor \frac{n-3}{4} \rfloor$. (5 points)
Exercise 6. Prove that $\mathrm{ex}(n, C_{13}) \leq 3n$. (10 points)
Exercise 7. Suggest a name for C_{13}! (5 points)

Library of $PTS(n)$ for 2, 3, and 4 blocks


A_1


A_2


B_1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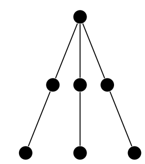
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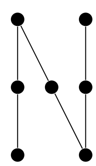
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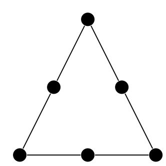
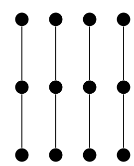
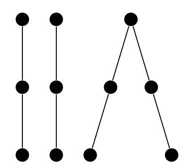

B_5


C_1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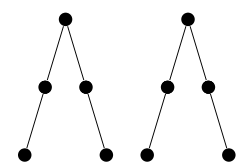
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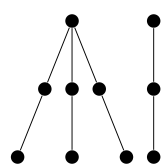
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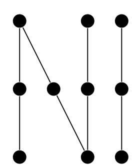
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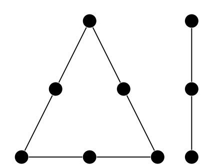
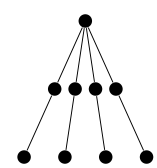

C_6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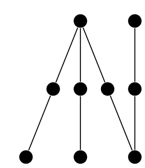
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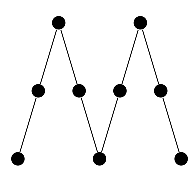
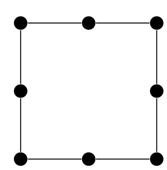
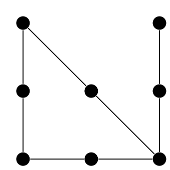
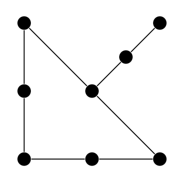
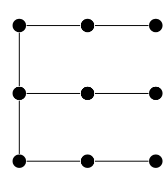
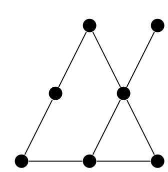

C_9

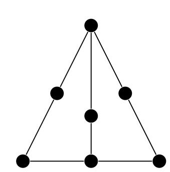

C_{10}

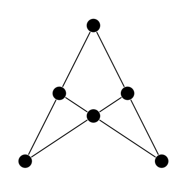

C_{11}

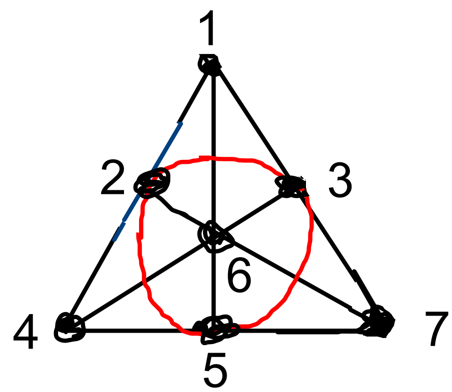

C_{12}


C_{13}

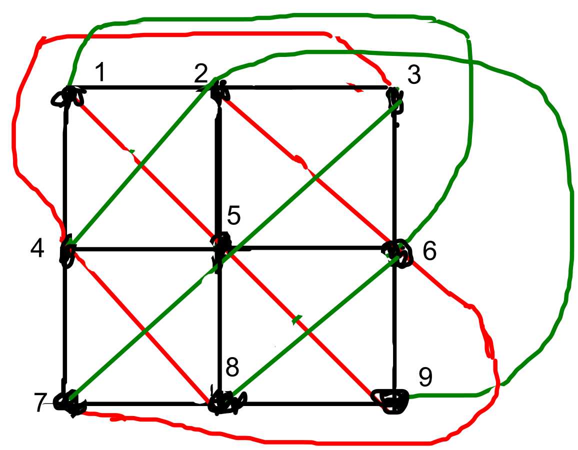

C_{14}


C_{15}


C_{16}



Fano plane



Affine plane