QUANTITATIVE HELLY-TYPE RESULTS

GERGELY AMBRUS

Convexity plays a central role in geometry as well as in computer science. In this project, we will focus on combinatorial aspects of convexity, by studying relaxations of Helly's theorem. Let us start with the central concepts.

Let C be a point set in $\mathbb{R}^d$. We say that C is *convex*, if along with any two points of it, C also contains the line segment between these two points. Equivalently, C is convex iff it is closed under taking finite *convex combinations*: for any $x_1, \dots, x_n \in C$, and any set of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for every i and $\sum_{i=1}^n \lambda_i = 1$, we have that

$$\sum_{i=1}^n \lambda_i x_i \in C\,.$$

Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be convex sets in $\mathbb{R}^d$. Helly's theorem, one of the cornerstones of combinatorial convexity, states the following:

Theorem 1. *If any $d+1$ sets of the family $\mathcal{C}$ share a common point, then there is a point common to all members of $\mathcal{C}$.*

The conclusion may also be written as $\cap\mathcal{C} \neq \emptyset$.

Helly's theorem has been generalized in many directions. There exist versions where convexity of the sets is not required, or where the ambient space differs from $\mathbb{R}^d$.

In the proposed research project, we will keep the assumption that the C_i are convex sets, which are all contained in $\mathbb{R}^d$. For a fixed family of such sets, we say that property $H(k)$ holds if the intersection of any k sets is non-empty. With that terminology, Helly's theorem states that $H(d+1)$ implies $H(n)$ for every $n \geq d+1$.

We are interested in making this statement *quantitative* in the following – broad – sense: assuming that the intersection of any k sets is *not small*, our goal will be to prove that the intersection of all the sets is not small either.

How to measure size? The first idea is to use *volume*. It turns out that it is not sufficient to assume that the intersection of any $d+1$ sets has large volume.

Introductory Problem 1. For $d \geq 3$ and for some $k \geq d+1$, give an example of a family of convex sets in $\mathbb{R}^d$ so that the intersection of any k of them has volume at least 1, yet, the intersection of all of them has arbitrarily small volume. Try to make k as large as possible.

In the above exercise, k may not be pushed beyond $2d-1$: it has been proved that if the intersection of any $2d$ sets has volume at least 1, then the intersection of all the sets has volume at least $(cd)^{-3d/2}$ with some constant c

(*you may also try to find a proof with any nontrivial lower bound*). It is conjectured that the right order of magnitude is $(cd)^{-d/2}$.

Research question 1. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a family of convex sets in $\mathbb{R}^d$ with the property that the intersection of any $2d$ members of $\mathcal{C}$ has volume at least 1. Prove that the volume of the intersection of all sets is at least $(cd)^{-\alpha d}$ with some constant c and $\alpha < 3/2$.

What other ways are there to measure the size of sets? One possibility is *diameter*. Again, in this case it is not sufficient to assume that the intersection of any $d+1$ sets has large diameter:

Introductory Problem 2. For $d \geq 3$ and for some $k \geq d+1$, give an example of a family of convex sets in $\mathbb{R}^d$ so that the intersection of any k of them has diameter at least 1, yet, the intersection of all of them is a single point. Try to make k as large as possible.

Yet, it is again sufficient to assume that the intersection of any $2d$ of the sets has diameter 1: it has very recently been proven that under that assumption, the diameter of the intersection of all the sets is at least $(2d)^{-3}$. It is conjectured that the truth is $c'd^{-1/2}$ with some constant $c' > 0$.

Research question 2. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a family of convex sets in $\mathbb{R}^d$ with the property that the intersection of any $2d$ members of $\mathcal{C}$ has diameter at least 1. Prove that the diameter of all the sets is at least $cd^{-\beta}$ with some constant c and $\beta < 3$.

There is a connection between the two problems: if a set has diameter 1, then its volume may be *very small* (zero, indeed). On the other hand, its volume may not be *too large*:

Introductory Problem 3. Assume that the planar convex set C has diameter 1. Give a nontrivial upper bound on the area of C. Aim to make this bound as strong as possible. Try to generalize it to higher dimensions.

A further possibility for measuring sets is that of *intrinsic volumes*, which are averages of the k-dimensional volumes of k-dimensional projections of a set. It is a natural question to extend the above results to these.

Research question 3. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a family of convex sets in $\mathbb{R}^d$ with the property that the kth intrinsic volume of the intersection of any $2d$ members of $\mathcal{C}$ is at least 1. Give a lower bound on the kth intrinsic volume of the intersection of all the sets.

Of course, we will spend the first half of the project learning all the necessary framework and methods. There are numerous directions for research here. The area has been very active in the last couple of years, which makes it an ideal opportunity to make a contribution which attracts interest.

If you are interested in participating the research project, please send your solutions to the above three Introductory Problems to the email address ambruge@gmail.com.