Groups and graph limits BSM research course warmup

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Let G be a connected, d-regular graph. For a vertex $v \in V(G)$ and natural n let $p_{v,2n}$ denote the probability that the random walk of length $2n$ starting at v returns to v.

Exercise 1 *Show that*

$$\lim_{n\to\infty} \sqrt[2n]{p_{v,2n}}$$

exists and is independent of v.

Let $l^2(G)$ denote the Hilbert space of square summable functions

$$l^2(G) = \left\{ f : V(G) \to \mathbb{R} \mid \sum_{x\in V(G)} f(x)^2 < \infty \right\}.$$

Let the Markov operator $M : l^2(G) \to l^2(G)$ be defined by

$$(Mf)(x) = \frac{1}{d} \sum_{(x,y)\in E(G)} f(y).$$

Exercise 2 *Let T_d denote the d-regular tree ($d \geq 2$). Show that there exists no $0 \neq f \in l^2(T_d)$ and $\lambda \neq 0$ such that $Mf = \lambda f$.*

Let S denote the unit circle on the plane with Lebesque measure λ and let r be an irrational rotation.

Exercise 3 *There exists no measurable subset $A \subseteq S$ such that $x \in A$ if and only if $r(x) \notin A$.*

Exercise 4 *For every $\varepsilon > 0$ there exists a measurable $A \subseteq S$ such that the above is ε-true, that is,*

$$\lambda\left(r(A) \cap A\right) + \lambda\left(r(A^c) \cap A^c\right) < \varepsilon.$$

Here A^c is the complement of A.

That is, the unit circle with an irrational rotation is not measurably bipartite but is almost measurably bipartite.

Exercise 5 *Let us define the random tree T_n on n points as follows. Let T_0 be a vertex with no edges. Take T_{n-1} and add a new vertex, attached to a uniform random vertex in T_{n-1}. What can you say about*

$$\lim_{n\to\infty} \frac{\text{\# of leaves of } T_n}{n}?$$