Forbidden Configurations

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April 21, 2021

We need some basic definitions. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Then an $m \times n$ simple matrix corresponds to a *simple hypergraph* or *set system* on m vertices with n edges as columns of the matrix are the characteristic vectors of sets in the set system. For a matrix A, let $|A|$ denote the number of columns in A. For a (0,1)-matrix F, we define that a (0,1)-matrix A has F as a *configuration* if there is a submatrix of A which is a row and/or column permutation of F, in notation $F \prec A$. Let $\mathrm{Avoid}(m, F)$ denote the set of all m-rowed simple matrices with no configuration F. The fundamental extremal problem is to compute

$$\mathrm{forb}(m, F) = \max_A \{|A| \colon A \in \mathrm{Avoid}(m, F)\}. \tag{1}$$

Let $\mathrm{Avoid}(m, \mathcal{F})$ denote the set of all m-rowed simple matrices with no configuration $F \in \mathcal{F}$. Define

$$\mathrm{forb}(m, \mathcal{F}) = \max_A \{|A| \ : \ A \in \mathrm{Avoid}(m, \mathcal{F})\}. \tag{2}$$

The following product is important. Let A and B be (0,1)-matrices. We define the product $A \times B$ by taking each column of A and putting it on top of every column of B. Hence if $|A| = a$ and $|B| = b$ then $|A \times B|$ is ab. For example, the vertex-edge incidence matrix of the complete bipartite graph $K_{m/2,m/2}$ is $I_{m/2} \times I_{m/2}$. Let I_m be the $m \times m$ identity matrix, I_m^c be the (0,1)-complement of I_m (all ones except for the diagonal) and let T_m be the triangular matrix, namely the (0,1)-matrix with a 1 in position i, j if and only if $i \leq j$. The following is the main motivating conjecture.

Conjecture 0.1 *[2] Let F be a $k \times \ell$ matrix with $F \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $X(F)$ denote the largest p such that there are choices $A_1, A_2, \ldots, A_p \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$ so that $F \not\prec A_1 \times A_2 \times \cdots \times A_p$. Then* $\mathrm{forb}(m, F) = \Theta(m^{X(F)})$.

Many special cases have been verified, for details one may consult [1]. In the current research proposal we aproach the problem from another direction. In extremal graph theory it is a classical question [3] that if a graph has more edges than allowed by a forbidden subgraph, then how many forbidden subgraphs are there? We ask the analogous question here. If matrix A has $\text{forb}(m, F) + k$ columns, then how many different configurations F are in A, as a function of k. The difference from classical extremal hypergraph heory is that configurations correspond to *induced* subhypergraphs. The question is mainly interesting for those configurations F for which $\text{forb}(m, F)$ is known exactly. For a start we could investigate the cases $F = I_2$ or $F = K_2$. We know that $\text{forb}(m, I_2) = \text{forb}(m, K_2) = m + 1$ What is the difference between the number I_2's and K_2's if our matrix has $m + 1 + k$ columns?

Another aspect is that how many columns do we need to have $k \times \ell$ configuration F on each k-tuples of rows?

Here are some **qualifying problems** to get into the mood:

1. Prove that $\text{forb}(m, F) = \text{forb}(m, F^c)$, where F^c is the $0-1$-complement of F.

2. What is $\text{forb}(m, I_2)$? What is $\text{forb}(m, \{I_2, T_2\})$?

3. Let F be a k-rowed matrix. Suppose we have $A \in \text{Avoid}(m, F)$ such that $|A| = \text{forb}(m, F)$. Consider deleting a row r. Let $C_r(A)$ be the matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. If we permute the rows of A so that r becomes the first row, then after some column permutations, A looks like this:
$$A = \begin{matrix} r \end{matrix}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r(A) & & C_r(A) & C_r(A) & & D_r(A) \end{bmatrix}. \tag{3}$$
where $B_r(A)$ are the columns that appear with a 0 on row r, but don't appear with a 1, and $D_r(A)$ are the columns that appear with a 1 but not a 0. Prove that
$$\text{forb}(m, F) \leq |C_r(A)| + \text{forb}(m-1, F). \tag{4}$$

4. Let K_k denote the $k \times 2^k$ simple $0-1$-matrix (configuration). Use the decomposition (3) and the inequality (4) to prove that $\text{forb}(m, K_k) = O(m^{k-1})$.

5. Prove that
$$\text{forb}(m, K_k) \geq \binom{m}{k-1} + \binom{m}{k-2} + \ldots + \binom{m}{0}. \tag{5}$$

6. Do we have equality in (5)?

7. Prove that
$$I_p \times T_p \in \text{Avoid}(m, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}). \tag{6}$$

References

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