Rigidity properties of braced triangulations

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Introduction

In this project we focus on *maximal planar graphs* (or *triangulations*) and *braced triangulations*, which are obtained by adding some extra edges to a triangulation. We shall study the rigidity properties of the *bar-and-joint frameworks* (or simply *frameworks*) whose graph is a (braced) triangulation.

A framework consists of rigid (fixed length) bars that meet at universal joints. The joints can move continuously in d-space so that the bar lengths and bar-joint incidences must be preserved. The framework is said to be *rigid* (in dimension d) if every such continuous motion results in a congruent framework, that is, the distance between each pair of joints is unchanged. The framework is said to be *globally rigid* if every other framework with the same graph and same bar lengths is congruent with it. Thus global rigidity implies rigidity.

In the *graph* of the framework vertices correspond to the joints and edges correspond to the bars. It is known that if the framework is in sufficiently general (generic) position then (global) rigidity depends only on its graph.

A celebrated result of A. Cauchy from the 19th century states that a triangulated convex polyhedron in three dimensions is rigid. Since the 1-skeleton of such a polyhedron is a triangulation, this result is our starting point. A recent research direction is to consider *braced polyhedra*, in which some additional edges are added to a tringulated convex polyhedron, and their graphs. It has been observed that they have stronger rigidity properties, especially if we assume that the connectivity of the graph is higher.

In this context connectivity refers to the vertex-connectivity of graph G: a graph G is said to be k*-connected*, for some positive integer k, if G has at least $k+1$ vertices and $G - X$ is connected for all subsets X of the vertex set of G with size less than k.

A related question that we might also consider is whether some of these triangulations are *unit ball graphs*, i.e. if they are the intersection graphs of an appropriate set of unit balls in three dimensions. Such unit ball graphs have further interesting rigidity properties.

Open problems

Rigidity theory is in the intersection of geometry, algebra, and combinatorics with several applications. It also includes quite a few questions concerning efficient algorithms. We shall chose an open problem suitable for the interested students. Three candidates are given below. (Each of these problems is defined is three dimensions.)

Problem 1 *Let G be a 5-connected braced triangulation with at least two bracing edges. Is it true that $G - \{e, f\}$ is rigid for every pair e, f of edges of G?*

A similar question is as follows.

Problem 2 *Let G be a 5-connected braced triangulation with at least two bracing edges. Is it true that $G - \{e\}$ is globally rigid for every edge e of G?*

We may also consider various extremal problems. For example:

Problem 3 *What is the minimum number of edges in a graph G on n vertices for which $G - v$ is globally rigid for all $v \in V(G)$? Can we construct an infinite family of extremal graphs which consists of braced triangulations?*

Basic definitions

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to the d-dimensional Euclidean space R^d. We consider the framework to be a straight line *realization* of G in R^d. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks (G, p) and (G, q) are *equivalent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $uv \in E$, where $dist(x, y)$ denotes the Euclidean distance between points x and y in R^d. Frameworks (G, p), (G, q) are *congruent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $u, v \in V$.

This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of R^d. We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p).

A *motion* (or *flex*) of (G, p) to (G, q) is a collection of continuous functions $M_v : [0, 1] \to R^d$, one for each vertex $v \in V$, that satisfy

$$M_v(0) = p(v) \text{ and } M_v(1) = q(v)$$

for all $v \in V$, and

$$dist(M_u(t), M_v(t)) = dist(p(u), p(v))$$

for all edges uv and for all $t \in [0, 1]$. The framework (G, p) is *rigid* if every motion takes it to a congruent framework (G, q).

Qualifying problems

Solve at least four out of the next five exercises.

Exercise 1. Show that every triangulation is 3-connected.

Exercise 2. Prove that the smallest 5-connected triangulation is the graph of the icosahedron.

Exercise 3. (a) Find an infinite family of 4-connected triangulations. (b) Find some 5-connected triangulations on more than 12 vertices.

Exercise 4. Describe the set of 6-connected triangulatios.

Exercise 5. Consider a triangulated convex polyhedron. As a bar-and-joint framework, it is rigid. After removing an edge (bar) it is no longer rigid. Illustrate this fact by analysing some examples, where you can describe the flex (continuous motion) of this non-rigid framework.

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