Regularity Based Spectral Clustering
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Qualifying exercises

The following graph based matrices will be used. Let G be a simple graph on vertex set $\{1,\dots,n\}$ with $n\times n$ adjacency matrix $\boldsymbol{A}=(a_{ij})$ (for $i\neq j$: $a_{ij}=a_{ji}=1$ if vertex i and j are connected and 0 otherwise; $a_{ii}=0$, $i=1,\dots,n$). Let $d_i=\sum_{j=1}^{n}$ be the degree of vertex i for $i=1,\dots,n$. We assume that G is connected, i.e., $d_i>0$, $i=1,\dots,n$. Let $\boldsymbol{D}$ denote the degree matrix: diagonal matrix containing $d_1,\dots,d_n$ in its main diagonal, and zeros otherwise.

The *Laplacian* matrix of G is $\boldsymbol{L}=\boldsymbol{D}-\boldsymbol{A}$. The *normalized (degree-corrected) adjacency* matrix of G is $\boldsymbol{A_D}=\boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$. The *normalized (degree-corrected) Laplacian* matrix of G is $\boldsymbol{L_D}=\boldsymbol{D}^{-1/2}\boldsymbol{L}\boldsymbol{D}^{-1/2}$.

Exercises to solve

1. Let K_n be the complete graph on n vertices (each $i\neq j$ pairs are connected). Find the eigenvalues together with corresponding eigenvectors of its $\boldsymbol{A},\boldsymbol{L},\boldsymbol{A_D},\boldsymbol{L_D}$ matrices.

   Please, derive those by hand calculations, numerical results without explanation are not accepted.

2. Let $K_{n,m}$ be the complete bipartite graph on $n+m$ vertices (there are two independent vertex classes with no edges inside on n and m vertices, respectively, and all possible edges between these two classes are present). Find the eigenvalues together with corresponding eigenvectors of the adjacency matrix $\boldsymbol{A}$ of $K_{n,m}$.

   Please, derive those by hand calculations, numerical results without explanation are not accepted.

3. Show that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary $n\times n$ symmetric, positive definite real matrices, then $\boldsymbol{AB}$ (usually not symmetric) has positive real eigenvalues. (Note that $\boldsymbol{A}$ and $\boldsymbol{B}$ usually do not commute, so their eigenvalues are not multiplied together.)