PIERCING THE CHESSBOARD

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Let C_n be the $n \times n$ chessboard in the plane: its cells are of the form $[i, i+1] \times [j, j+1]$ where $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$. The main question we are going to study is the following intuitive problem.

How many lines are needed in order to pierce (intersect) every cell of C_n in the interior?

Let l_n denote the minimal cardinality of such a piercing set. For example, $l_1 = 1$ and $l_2 = 2$. Note that if we require only *having a common point* with each cell rather than piercing it, then 1 line suffices for $n = 2$ as well.

Estimating l_n is simple if one doesn't aim for the best possible estimates. The first introductory problem addresses this question.

Introductory Problem 1. Show that $l_n \leq n$ for each $n \geq 1$.

Introductory Problem 2. Show that $l_n \geq \frac{n}{2}$ for each $n \geq 1$.

It turns out that these bounds are not sharp.

Introductory Problem 3. Show that the above bounds on l_n are not sharp, for as many values of n as you can.

Our goal in this project will be to improve on the existing bounds for l_n. Even though the problem seems to be independent, the situation is different: this is a rather nice discrete version of the famous *plank problem*, which original version states that if one wishes to cover the unit disc using *planks*, that is, regions of the plane between two parallel lines, then the total width of these planks must be at least as large as the diameter of the disc. A very elegant proof for this statement may be derived using the following lemma. Let S^2 be the unit sphere in 3 dimensions.

Introductory Problem 4. Consider two parallel planes at distance t which both intersect S^2. Take the intersection of the sphere S^2 with region between the two planes – this is called a *spherical zone.* Depending on the location of the planes, calculate the surface area of a spherical zone whose bounding planes are at distance t apart.

Once you found out the (surprising) answer, you are able to prove the original plank problem.

Introductory Problem 5. Using you answer to Problem 4, prove that any system of planks covering the unit disc in the plane must have total width at least 2.

We will study the connection between the plank problem and the chessboard piercing problem in detail. En route, we are going to see many generalizations and alternate questions. We will address the 3-dimensional version of the question as well as its modification to different piercing sets.

Prerequisites are not needed, as we are going to begin the project with learning all the necessary tools. Yet, a good geometric intuition, and a zestful interest in doing a research project is mandatory! :)

If you are interested in participating the research project, please send your solutions to the above five Introductory Problems (or as many as you are able to solve) to the email address ambruge@gmail.com.