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Note

On r-Cover-free Families

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A very short proof is presented for the almost best upper bound for the size of an r-cover-free family over n elements. © 1996 Academic Press, Inc.

A family of sets $\mathscr{F}$ is called *r-cover-free* if $A_0 \not\subseteq A_1 \cup A_2 \cup \cdots \cup A_r$ holds for all distinct $A_0, A_1, ..., A_r \in \mathscr{F}$. Let $T(n, r)$ denote the maximum cardinality of such an $\mathscr{F}$ over an n-element underlying set. This notion was introduced by Kautz and Singleton [4] in 1964 concerning binary codes. They proved $\Omega(1/r^2) \leqslant \log T(n, r)/n \leqslant O(1/r)$ (log is always of base 2). This result was rediscovered several times in information theory, in combinatorics [2], and in group testing [3]. A recent account and related problems can be found in Körner [5]. Dyachkov and Rykov [1] obtained with a rather involved proof that $\log T(n/r)/n \leqslant O(\log r/r^2)$. Recently, Ruszinkó [6] gave a purely combinatorial proof. Our aim is to present an even simpler argument to show that

$$\frac{\log T(n, r)}{n} \leqslant \frac{4 \log r + O(1)}{r^2}. \qquad (1)$$

This upper bound is twice as good as that of [6], but about half as good as that obtained from the inductive proof of [1]. Our argument is implicitly contained in Erdős, Frankl, and Füredi [2].

THEOREM. *If $\mathscr{F}$ is a family of subsets of an n-element underlying set V such that no set $F_0 \in \mathscr{F}$ is contained in the union of r other members of $\mathscr{F}$, then*

$$|\mathscr{F}| \leqslant r + \binom{n}{\left\lceil (n-r) \Big/ \binom{r+1}{2} \right\rceil}. \qquad (2)$$

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Proof. Fix an integer t with $n/2 \geqslant t > 0$. Define $\mathscr{F}_t \subset \mathscr{F}$ as the family of members having its *own* t-subset, i.e., $\mathscr{F}_t := \{F \in \mathscr{F}$: there exists a t-element set $A \subseteq F$ such that $A \nsubseteq F'$ holds for every other $F' \in \mathscr{F}\}$, and let $\mathscr{A}$ be the family of these t-subsets. Let $\mathscr{F}_0 := \{F \in \mathscr{F} : |F| < t\}$, and let $\mathscr{B}$ be the family of t-sets containing a member of $\mathscr{F}_0$, i.e., $\mathscr{B} := \{T : T \subset V,\ |T| = t$, and there exists some $F \in \mathscr{F}_0$ with $T \supset F\}$. The set-system $\mathscr{F}$ is an *antichain*; no two members contain each other. This implies that $\mathscr{A}$ and $\mathscr{B}$ are disjoint. A lemma of Sperner [7] states that $|\mathscr{F}_0| \leqslant |\mathscr{B}|$. We obtain that $|\mathscr{F}_0 \cup \mathscr{F}_t| \leqslant |\mathscr{A}| + |\mathscr{B}| \leqslant \binom{n}{t}$.

Let $\mathscr{F}' := \mathscr{F} \setminus (\mathscr{F}_0 \cup \mathscr{F}_t)$. We claim that $F \in \mathscr{F}'$, $F_1, F_2, ..., F_i \in \mathscr{F}$ imply

$$\left| F \Big\backslash \bigcup_{j \leqslant i} F_j \right| > t(r-i). \tag{3}$$

Indeed, if $F \setminus (F_1 \cup \cdots \cup F_i)$ can be written as the union of the t-element sets $A_{i+1}, A_{i+2}, ..., A_r$, then by the choice of F there are members $F \neq F_j \in \mathscr{F}$ with $A_j \subseteq F_j$. Therefore $F \subset (F_1 \cup \cdots \cup F_r)$, a contradiction.

Inequality (3) implies that for $F_0, F_1, ..., F_r \in \mathscr{F}'$ one has $|\bigcup_{i \leqslant r} F_i| = |F_0| + |F_1 \setminus F_0| + |F_2 \setminus (F_1 \cup F_0)| + \cdots + |F_r \setminus (F_0 \cup F_1 \cup \cdots \cup F_{r-1})| \geqslant r + 1 + t\binom{r+1}{2}$. Here the right-hand side exceeds n for $t := \lceil (n-r)/\binom{r+1}{2} \rceil$, implying $|\mathscr{F}'| \leqslant r$. ∎

Finally, the upper bound (1) easily follows from (2) using $\binom{n}{t} \leqslant n^t/t! < (en/t)^t$.

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