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Note

On the Upper Bound of the Size of the r-Cover-Free Families*

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Let $T(r, n)$ denote the maximum number of subsets of an n-set satisfying the condition in the title. It is proved in a purely combinatorial way that for n sufficiently large

$$\frac{\log_2 T(r, n)}{n} \leqslant 8 \cdot \frac{\log_2 r}{r^2},$$

holds.

1. INTRODUCTION

The notion of the r-cover-free families was introduced by Kautz and Singleton in 1964 [17]. They initiated investigating binary codes with the property that the disjunction of any $\leqslant r$ $(r \geqslant 2)$ codewords are distinct (UD_r codes). This led them to studying the binary codes with the property that none of the codewords is covered by the disjunction of $\leqslant r$ others (superimposed codes, ZFD_r codes; P. Erdős, P. Frankl and Z. Füredi called the corresponding set system r-cover-free in [7]).

Since then many results have been proved about the maximum size of these codes. Various authors studied these problems from basically three different points of view, and these three lines of investigations were almost independent of each other. This is why many results were found first

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in information theory ([1, 4, 5, 14–17]), were later rediscovered in combinatorics ([2, 6, 7, 10, 18, 19]), or in group testing ([12, 13]), and vice versa.

We approach this area from the combinatorial side. Our main goal is to estimate the maximal size of the family of subsets of an n-element set with the property that no set is covered by the union of r others.

2. Notation and Definitions

Let S be an n-element set. The set of all subsets of S is denoted 2^S. $\binom{S}{k}$ denotes the set of all k-subsets of S ($k \geqslant 0$). If $|S| = n$, then $|\binom{S}{k}| = \binom{n}{k}$. We denote by $[n]$ the set $\{1, 2, ..., n\}$, and $\log x$ is always of base 2. A set system $\mathscr{A} \subseteq 2^S$ is called k-uniform if its members are k-sets. It is usually supposed that the underlying set of the set systems is $[n]$.

We call $\mathscr{F}' \subset 2^S$ r-distinct, if

$$\bigcup_{i=1}^{k} A_i \neq \bigcup_{j=1}^{l} B_i$$

for any

$$\{A_1, A_2, ..., A_k\} \neq \{B_1, B_2, ..., B_l\},$$

$1 \leqslant k, l \leqslant r$; $A_1, A_2, ..., A_k, B_1, B_2, ..., B_l \in \mathscr{F}'$. $\mathscr{F} \subset 2^S$ is r-cover-free, if

$$A_0 \not\subseteq A_1 \cup A_2 \cup \cdots \cup A_r$$

holds for all distinct $A_0, A_1, ..., A_r \in \mathscr{F}$. $\mathscr{F}^* \subset 2^S$ is $<r$ part intersecting, if

$$|A_i \cap A_j| < \frac{1}{r} \min\{|A_i|, |A_j|\}$$

for any distinct $A_i, A_j \in \mathscr{F}^*$ holds. We denote by $T'(r, n)$, $T(r, n)$, $T^*(r, n)$ and $T'(r, n, k)$, $T(r, n, k)$, $T^*(r, n, k)$ the maximum cardinality of the corresponding set systems in the general and in the k-uniform cases, respectively. We provide upper bounds on these functions for r fixed and n tending to infinity. We call

$$R(r) = \limsup_{n \to \infty} \frac{\log T(r, n)}{n}$$

the *rate* of the r-cover-free family. The following proposition is obvious from the definitions.

PROPOSITION 2.1. *If $\mathscr{F}$ is $<r$ part intersecting, then $\mathscr{F}$ is r-cover-free; and if $\mathscr{F}$ is r-cover-free, then $\mathscr{F}$ is r-distinct. Hence*

$$T^*(r, n) \leqslant T(r, n) \leqslant T'(r, n), \quad \text{and} \quad T^*(r, n, k) \leqslant T(r, n, k) \leqslant T'(r, n, k).$$

The following upper and lower bounds were proved in [1, 4, 5, 7, 13]: there exist two (absolute) constants c_1, c_2 such that

$$\frac{c_1}{r^2} \leqslant \frac{\log T(r, n)}{n} \leqslant \frac{c_2}{r} \tag{1}$$

for any n. In most papers the lower bound is proved by probabilistic methods. In [13] Hwang and Sós used a greedy-type algorithm to generate $<r$ part intersecting families for proving the lower bound. The upper bound was proved using the observation that, by definition, $\sum_{i=1}^{r} \binom{T'}{i} \leqslant 2^n$. The gap between the upper and lower bounds is rather large. Dyachkov and Rykov obtained a better upper bound [4],

$$\frac{\log T(r, n)}{n} \leqslant c_3 \frac{\log r}{r^2} \tag{2}$$

for some absolute constant c_3 and any n. Their proof is rather involved. Here we give a simple and purely combinatorial proof of this result.

3. THE UPPER BOUND

First we have to prove some lemmata.

LEMMA 3.1. *If r divides k then*

$$T(r, n, k) \leqslant r \left(\binom{n}{\frac{k}{r}} \bigg/ \binom{k}{\frac{k}{r}} \right) \tag{3}$$

Proof. Let $\mathscr{F}$ be a k-uniform r-cover-free family and let r divide k. Let A_0 be an arbitrary element of $\mathscr{F}$. By Baranyai's Theorem [3] (asserting that if r divides k then the r-uniform complete hypergraph on k vertices has one-factorisation) one can list all subsets of size k/r of A_0 in the following way:

(a) in each row there is a partition of A_0;

(b) each subset is listed only once.

So each row contains r subsets and the number of rows is $s=\binom{k}{k/r}/r$. This family of partitions can be represented in the matrix form

$$\begin{matrix} B_{1,1} & B_{1,2} & \cdots & B_{1,r} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s,1} & B_{s,2} & \cdots & B_{s,r}, \end{matrix}$$

where (a) means that $\bigcup_{j=1}^{r} B_{i,j}=A_0$ for any $1\leqslant i\leqslant s$ and (b) means that $B_{i,j}\neq B_{k,l}$ for any $(i,j)\neq(k,l)$.

For any $1\leqslant i\leqslant s$ the ith row contains at least one set $B_{i,j}$ which is not contained in any other $A_{v_j}\in\mathscr{F}$. Indeed, otherwise for some i we could find A_{v_j} $(j=1,\ldots,r)$ that $B_{i,j}\subseteq A_{v_j}\neq A_0$, and therefore

$$A_0=B_{i,1}\cup B_{i,2}\cup\cdots\cup B_{i,r}\subseteq A_{v_1}\cup A_{v_2}\cup\cdots\cup A_{v_r},$$

which would violate the condition.

Hence A_0 has at least s subsets of size k/r, which are not contained in any other $A_j\in\mathscr{F}$ (*own subsets*). Since the number of all subsets of the underlying set is $\binom{n}{k/r}$, we get that

$$|\mathscr{F}|\,s=|\mathscr{F}|\binom{k}{\frac{k}{r}}\Big/r\leqslant\binom{n}{\frac{k}{r}}. \quad ∎$$

From Lemma 3.1 the following stronger version of the lemma from [4] follows trivially.

LEMMA 3.2.

$$T(r,n,k)\leqslant r\binom{n}{\left\lceil\frac{k}{r}\right\rceil}\Bigg/\binom{\left\lfloor\frac{k}{r}\right\rfloor r}{\left\lfloor\frac{k}{r}\right\rfloor}. \tag{4}$$

Remark 1. Proposition 2.1 of Erdős *et al.* [7] asserts that if we set $t=\lceil k/r\rceil$ then

$$T(r,n,k)\leqslant\binom{n}{t}\Big/\binom{k-1}{t-1}. \tag{$*$}$$

This result was obtained by using the following

LEMMA 1F (Frankl [8]). *If $\mathscr{F}$ is a family of t-subsets of a k-set, and for any r sets*

$$F_1, \ldots, F_r \in \mathscr{F} \qquad \bigcap_{j=1}^{r} F_j \neq \emptyset,$$

if $rt/(r-1) \leqslant k$, then $|\mathscr{F}| \leqslant \binom{k-1}{t-1}$.

In the case when r does not divide k, $(*)$ is stronger than (4), but since it does not give a better exponent for $T(r, n, k)$, we use the inequality of Lemma 3.1.

It is also worth mentioning that in [9, Theorem 3.4] Frankl and Füredi proved that for fixed k and r

$$\lim_{n \to \infty} T(r, n, k) \bigg/ \binom{n}{t}$$

exists and equals $(\binom{k}{t} - m(k, t, l))^{-1}$, where $l = k - r(t-1) - 1$. (For the definition of $m(k, t, l)$ see [9].)

Obviously

$$T(r, n) \leqslant n \cdot \max_{1 \leqslant k \leqslant n} T(r, n, k) \tag{5}$$

and by (1) $T(r, n)$ is exponential in n. So the factor n is insignificant in (5). This leads to the following question: for which k does $T(r, n, k)$ attains its maximum? If we knew this, we could estimate $T(r, n)$ from (4). But for $k = n/2$, (4) yields only

$$\frac{\log T(r, n, n/2)}{2} \leqslant \frac{c}{r}$$

(for some constant c).

PROPOSITION 3.1. $T(r, n) \leqslant T(r, 2n, n)$.

Proof. Let $\mathscr{F} = \{A_1, \ldots, A_T\}$, $A_i \subseteq [n]$. For $i = 1, \ldots, T$ let $B_i = \{x + n\colon x \in \bar{A}_i\}$ ($B_i \subseteq \{n+1, \ldots, 2n\}$), $C_i = A_i \cup B_i$ ($C_i \subseteq [2n]$), and $\mathscr{F}^u = \{C_1, \ldots, C_T\}$. $\mathscr{F}^u$ is an n-uniform r-cover-free family on a $2n$ element set, $|\mathscr{F}^u| = |\mathscr{F}|$. ∎

By Proposition 3.1,

$$R(r) \leqslant 2 \cdot \limsup_{n \to \infty} \frac{\log T(r, 2n, n)}{2n},$$

so the size of the $n/2$ uniform r-cover-free families can be very large. (The rate is at most two times lower than in the nonuniform case.) Thus, if we want to get a better upper bound for $(\log T(r, n))/n$ than c/r, it is not enough to use the inequality (5). We have to compress somehow the elements of $\mathscr{F}$ without losing the r-cover-free property, since for smaller k (5) gives a better found.

LEMMA 3.3. *Let A_i be an arbitrary element of $\mathscr{F} = \{A_1, A_2, ..., A_T\}$ and $B_i \subseteq A_i$ an arbitrary subset of A_i. If $\mathscr{F}$ is r-cover-free, then*

(a) $\mathscr{F}^1 = \{A_j \backslash B_i\}_{j=1,...,T}^{j \neq i}$ *is $(r-1)$-cover-free,*

(b) $|\mathscr{F}^1| = T-1$.

Proof. (a) Suppose that

$$A_{j_0} \backslash B_i \subseteq (A_{j_1} \backslash B_i) \cup (A_{j_2} \backslash B_i) \cup \cdots \cup (A_{j_{r-1}} \backslash B_i)$$

for some $\{j_0, j_1, ..., j_{r-1}\} \subseteq [T] \backslash \{i\}$. Then

$$A_{j_0} \subseteq A_{j_1} \cup A_{j_2} \cup \cdots \cup A_{j_{r-1}} \cup A_i,$$

which is a contradiction.

(b_1) For any $A_j \neq A_i$ we have $A_j \nsubseteq A_i$, so we threw out only A_i from $\mathscr{F}$.

(b_2) $A_k \backslash B_i \neq A_l \backslash B_i$ if $k \neq l$, so we did not merge any two distinct members of $\mathscr{F}$. Indeed, suppose that $A_k \backslash B_i = A_l \backslash B_i$ and $k \neq l$. Then $A_k \subseteq A_l \cup A_i$, which is a contradiction ($r \geqslant 2$). ∎

THEOREM 3.1.

$$\frac{\log T(r, n)}{n} \leqslant 8 \cdot \frac{\log r}{r^2}. \tag{6}$$

Proof. First we assume that r^2 divides n and n/r is even. Let $\mathscr{F}$ be an arbitrary r-cover-free family. We use the following *set compression algorithm.*

(1) $\mathscr{F}^0 = \mathscr{F}$.

(2) If every element of $\mathscr{F}^i$ is of size $\leqslant 2n/r$, then $\tilde{\mathscr{F}} = \mathscr{F}^i$. If $\mathscr{F}^i = \{A_1^{(i)}, A_2^{(i)}, ..., A_{T-i}^{(i)}\}$ contains a set $A_{j_0}^{(i)}$ of size $> 2n/r$, then put $\mathscr{F}^{i+1} = \{A_j^{(i)} \backslash A_{j_0}^{(i)}\}_{j=1,...,T-i}^{j_0 \neq j}$.

In each step of this algorithm we throw out more than $2n/r$ elements. Since the underlying set of $\mathscr{F}$ is of size n, our algorithm will stop in at most $r/2$ steps. Suppose that during this algorithm we threw out p elements from the underlying set in q steps. Let $T(r, n, \leqslant k)$ denote the maximum cardinality of an r-cover-free family of subsets of $[n]$ of size $\leqslant k$. Then, by Lemma 3.3 and set compression algorithm,

$$\begin{aligned}
T(r, n) &\leqslant T\left(r-q, n-p, \leqslant \frac{2n}{r}\right)+q \leqslant T\left(\frac{r}{2}, n, \leqslant \frac{2n}{r}\right)+\frac{r}{2} \\
&\leqslant \sum_{k=1}^{2n/r} T\left(\frac{r}{2}, n, k\right)+\frac{r}{2} \leqslant \sum_{k=1}^{2n/r}\left(\frac{r}{2}\binom{n}{\frac{2k}{r}}\Big/\binom{k}{\frac{2k}{r}}\right)+\frac{r}{2} \\
&\leqslant \sum_{k=1}^{2n/r} \frac{r}{2} \cdot \binom{n}{\frac{2k}{r}} \leqslant n \cdot \binom{n}{\frac{4n}{r^2}}.
\end{aligned}$$

Let $h(x)$ $(0<x<1)$ be the binary entropy function; that is,

$$h(x)=x \log \frac{1}{x}+(1-x) \log \frac{1}{1-x}.$$

Then, by [11],

$$\binom{n}{cn} \leqslant 2^{n \cdot h(c)} \cdot n^{k_1},$$

where k_1 is an absolute constant. Therefore

$$\begin{aligned}
\log T(r, n) &\leqslant (k_1+1) \cdot \log n+n \cdot h\left(\frac{4}{r^2}\right) \\
&=o(n)+n\left(\frac{4}{r^2} \log \frac{r^2}{4}+\left(1-\frac{4}{r^2}\right) \log \left(\frac{r^2}{r^2-4}\right)\right) \\
&=o(n)+n\left(8 \frac{\log r}{r^2}-\frac{8}{r^2}+\frac{1}{r^2} \log \left(1+\frac{4}{r^2-4}\right)^{r^2-4}\right) \\
&\leqslant o(n)+n\left(8 \frac{\log r}{r^2}-\frac{8}{r^2}+\frac{4 \log e}{r^2}\right) \\
&\leqslant o(n)+8 \frac{\log r}{r^2} \cdot n.
\end{aligned}$$

If r^2 does not divide n or r is odd, the same proof works; we only have to be more careful with the integer parts.

Remark 2. A more careful computation would give a better constant than of 8.

4. Final Remarks

The most important thing would be to narrow the gap between the upper and lower bounds on $T(r, n)$, (see, e.g., Hwang and Sós [13]). Of course, the same question applies also to $T^*(r, n)$, $T(r, n)$, and $T'(r, n)$. In [17] one can find the proof of the following theorem.

If $\mathscr{F}'$ is r-distinct then it is $(r-1)$-cover-free.

So by this theorem and Proposition 2.1 $T(r, n)$ and $T'(r, n)$ are very closed. On the other hand, if $k \geqslant n/r$, then by Johnson's second bound [16] $T^*(r, n, k)$ is only polynomial in n. The proof of the lower bound suggests that it attains the maximum in k at about $n/(r+1)$. By the *set compression algorithm* we suppose that $T(r, n, k)$ attains the maximum in k about n/r, too. This suggests that $T(r, n)$ and $T^*(r, n)$ are neither significantly different.

We consider the following problem. The estimations in the proof of Theorem 3.1 are very loose. Is it possible to prove the $R(r) = o(\log r/r^2)$ upper bound using the set compression algorithm?

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