RESEARCH PROBLEM: HELLY-TYPE THEOREMS FOR ELLIPSES

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Helly's theorem states that if for a finite family of convex sets in $\mathbb{R}^d$, any $d+1$ members have a non-empty intersection, then the family has a non-empty intersection.

This result is the starting point of a range of exciting geometric and combinatorial questions on intersection patterns of convex sets. One particular direction is looking at quantitative versions, where instead of requiring the intersection to be non-empty (contain a point), we consider intersections of a certain volume, or those that contain an ellipsoid of a certain volume. Here is a *colorful* example.

Theorem 1 (Colorful Helly-type theorem for ellipsoids)**.** *Let $\mathcal{C}_1, \dots, \mathcal{C}_{3d}$ be finite families (color classes) of convex bodies in $\mathbb{R}^d$. Assume that for any colorful choice $C_1 \in \mathcal{C}_1, \dots C_{3d} \in \mathcal{C}_{3d}$ of $3d$ sets, the intersection $\bigcap_{i=1}^{3d} C_i$ contains an ellipsoid of volume at least 1.*

Then there is an $1 \le i \le 3d$ such that $\bigcap_{C \in \mathcal{C}_i} C$ contains an ellipsoid of volume at least d^{-10d^2}.

One particular **open problem** is the following question on the plane.

Does the above theorem hold for $d = 2$ when the number of color classes is not 6 ($= 3d$), but 5 or 4?

Practice questions. *It is not necessary to solve all of them to participate.*

(1) What is the largest size of a set of pairwise orthogonal unit vectors in $\mathbb{R}^d$?

(2) Let $\mathbf{n}$ and $\mathbf{v}$ be non-zero vectors in $\mathbb{R}^d$. It is known that $\mathbf{v}$ can be written as $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_o$, where $\mathbf{v}_p$ is parallel with $\mathbf{n}$, and $\mathbf{v}_o$ is orthogonal to $\mathbf{n}$. How do you compute $\mathbf{v}_p$ and $\mathbf{v}_o$? *Hint:* You can use scalar product.

(3) What is the largest size of a set of unit vectors in $\mathbb{R}^d$ where the angle of any two vectors is obtuse (greater than $\pi/2$)?

(4) Let $K_1, \dots, K_n$ be compact (closed and bounded) convex sets in $\mathbb{R}^d$ with $\bigcap_{i=1}^n K_i = \emptyset$. Show that there are halfspaces $H_1, \dots, H_n$ such that $K_1 \subset H_1, \dots, K_n \subset H_n$ with $\bigcap_{i=1}^n H_i = \emptyset$. *Hint:* Do you know this statement for $n = 2$? If not, can you prove it (not obvious)? Can you deduce the general case from the $n = 2$ case?

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