PEELING THE ONION – THE CONVEX LAYER PROCESS

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This rather unusual title is merged from two already published articles: " Counting the onion" by Ketan Dalal, published in 2004 by Random Structures and Algorithms; and "Peeling the grid" by Sariel Har-Peled and Bernard Lidický, published in the SIAM Journal of Discrete Mathematics, back in 2013. So what does it stand for?

Let X be a finite set of points in $\mathbb{R}^d$. The *convex hull* of X is a polytope with some finite number of vertices – these are points of X, called the *extreme points* of X. In case you are not familiar with this notion – besides checking Google –, just imagine a balloon containing all the points. Once you pierce the balloon (well, not too brutally), it will shrink immediately, and – with good approximation – it is going to be suspended by some of the points, which are exactly the extreme points of X.

In the current setting, the set of extreme points of X is referred to as the *convex layer* of X, in notation, $V(X)$. We can also call it the *first* convex layer.

Now, imagine that we drop the points of $V(X)$ from X, and call the remaining set X_1. For convenience, let $X_0 = X$. Clearly, $X_1 \subset X_0$.

Why stop here? We may replace X_0 by X_1, and repeat the process: take the extreme points of X_1 (that is, the convex layer), and delete them from X_1. The remaining set shall be called X_2. And so on! Thus, we may define recursively the sets X_i as

$$X_i = X_{i-1} \setminus V(X_{i-1})$$

for $i = 1, 2, \ldots$. Clearly, this is a nested sequence: $X_i \subset X_{i-1}$. It is also easy to see, that after finitely many steps, we arrive at the empty set (why?). This process is called the *peeling process*.

We are interested in the number of steps the peeling process takes to terminate (i.e. to reach the empty set), given the set X. This number is called the *layer number* of X, denoted as $L(X)$. We want to give *asymptotic estimates* for $L(X)$ in terms of the number of points in X, that we are going to denote by n.

Introductory Problem 1. Give an upper bound on $L(X)$ that is valid for every set X consisting of n points. Try to give the best possible bound, and show that it is tight. Does this bound depend on the dimension of the ambient space? If yes, what is its relation to d? If not, how could we put an extra condition on X that somehow forces the dimension to play a role?

Of course, the layer number is related intimately to the structure of X. We expect different behaviours for different sets. For example, take a very regular discrete point set in the plane: the $\sqrt{n}$-by-$\sqrt{n}$ square grid (here we assume that n is a square). This has n points. It was proven in [2] that the layer number of the grid is proportional to $n^{2/3}$ (for the ones familiar with computer science, its order of magnitude is $\Theta(n^{2/3})$). One goal of the research will be trying to extend this result to higher dimensions:

Open research question 1. Determine the order of magnitude of the layer number of the d-dimensional square grid.

At least, we will try to tackle the 3–dimensional case.

On the other hand, uniform random point sets have also been studied: here, we select n random, uniform points in the unit disk – this serves as our set X. It was proven in [1] that in this case, if $|X| = n$, then $L(X) = \Theta(n^{2/3})$), thus, it has the same order of magnitude as the case of the regular grid. In the random setting, the higher dimensional cases have also been studied, and it was shown that if the points are chosen randomly and uniformly from the d-dimensional unit ball B^d, then $L(X) = \Theta(n^{2/(d+1)})$. Our second goal will be to try to extend this result for other probability distributions.

Open research question 2. Determine the order of magnitude of the layer number of an n-element random set, where the points are chosen according to the normal distribution in the plane, or in $\mathbb{R}^d$.

Finally, instead of random point sets, one may impose other conditions on the set X. In [3], the authors consider a regularity condition on X, and give bounds on $L(X)$. We will try to push the frontiers further in that direction.

Introductory Problem 2. For any given $k \geq 1$, construct a set in the plane with peeling number k and cardinality $4k$. Construct another set with peeling number 4 and cardinality $4k$. Can you construct a set with peeling number $\sqrt{k}$ and cardinality k (assuming k is a square)?

In case you are interested in this research project, please address the two Introductory Problems above, and send your solutions to my email address `ambruge@gmail.com` no later than the Welcome Party. If you prefer scribbling, you can also just hand me your work at the Welcome Party.

Prerequisites: some geometry and combinatory knowledge would be preferred, but we will cover the topics needed.

References

[1] Ketan Dalal, *Counting the onion.* Random Structures Algorithms **24** (2004), no. 2., 155–165.

[2] Sariel Har-Peled and Bernard Lidickỳ, *Peeling the grid.* SIAM J. Disc. Math. **27**(2013), no. 2., 650–655.

[3] Ilkyoo Choi, Weonyoung Joo, and Minki Kim, *The layer number of α-evenly distributed point sets.* Manuscript, 2019.

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