TRANSVERSAL PROPERTIES OF FAMILIES OF CONVEX SETS

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Convexity plays a central role in geometry as well as in computer science. In this project, we will focus on combinatorial aspects of convexity, by studying relaxations of Helly's theorem. Let us start with the central concepts.

Let C be a point set in $\mathbb{R}^d$. We say that C is *convex*, if along with any two points of it, C also contains the line segment between these two points. Equivalently, C is convex iff it is closed under taking finite *convex combinations*: for any $x_1, \dots, x_n \in C$, and any set of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for every i and $\sum_{i=1}^n \lambda_i = 1$, we have that

$$\sum_{i=1}^{n} \lambda_i x_i \in C\,.$$

Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be convex sets in $\mathbb{R}^d$. Helly's theorem, one of the cornerstones of combinatorial convexity, states the following:

Theorem 1. *If any $d+1$ sets of the family $\mathcal{C}$ share a common point, then there is a point common to all members of $\mathcal{C}$.*

The conclusion may also be written as $\cap\, \mathcal{C} \neq \emptyset$.

Helly's theorem has been generalized in many directions. There exist versions where convexity of the sets is not required, or where the ambient space differs from $\mathbb{R}^d$.

In the proposed research project, we will keep the assumption that the C_i are convex sets, which are all contained in $\mathbb{R}^d$. The point where we relax the original condition is that we require that *any k sets have a point in common*, with some $k \leq d$. We are going to study special cases of the following general question.

Research question 1. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a family of convex sets in $\mathbb{R}^d$ with the property that any k members of $\mathcal{C}$ share a common point. What may we say about the structure of $\mathcal{C}$?

It is easy to see that members of $\mathcal{C}$ no longer necessarily share a common point:

Qualifying Problem 1. For any $d \geq 1$, give an example of finitely many convex sets in $\mathbb{R}^d$ so that any d of them have non-empty intersection, but no $d+1$ of them share a common point.

Therefore, such a strong intersection property does not hold. Yet, there are some positive results: we may relax the notion of a *common point*. First, we replace points by *lines*, that is, 1 dimensional affine subspaces. The line

ℓ is called a *transversal* to the family $\mathcal{C}$, if it hits every member of $\mathcal{C}$, that is, $\ell \cap C_i \neq \emptyset$ for every $C_i \in \mathcal{C}$.

Qualifying Problem 2. For any $d \geq 2$, prove the following statement. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a finite set of convex sets in $\mathbb{R}^d$, of which any d have a non-empty intersection. Prove that there exists a common transversal to $\mathcal{C}$, that is, a line which hits every member of $\mathcal{C}$.

The next step is decrease the number of intersecting sets by 1. Surprisingly, we already find ourselves in virgin territory! The easiest open question is the following.

Research question 2. Assume that $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ is a family of convex sets in $\mathbb{R}^3$ with the property that any 2 members of $\mathcal{C}$ share a common point. Does there exist a line which hits a positive proportion of the members of $\mathcal{C}$ (where the proportion is independent of n)?

That we may only look for a partial transversal is showed by the following exercise.

Qualifying Problem 3. Give an example of a finite family of convex sets in $\mathbb{R}^3$ so that any two of them share a common point, but there is no common line transversal to all of them.

Although some background in convex geometry and combinatorics is helpful, we will start the semester with learning all the necessary tools. Thus, it is going to be an excellent opportunity to study convexity, apply combinatorial ideas, and possibly use analytical and probabilistic tools.

If you are interested in participating in this research project, please send your solutions to the above three Qualifying Problems to the email address ambruge@gmail.com by August 15th.