COURSE DESCRIPTION FOR "EXPLICIT BURGESS TYPE ESTIMATES FOR COMPOSITE MODULI"

1. The topic

Recently, a question about explicit versions of the Burgess bound for character sums in the case of composite moduli has been raised on mathoverflow. The goal of the research class is to work out such an explicit bound. Prerequisites: basics of number theory, characters of finite abelian groups, analytic techniques.

2. Preliminary assignment

I expect the solutions of these exercises not later than the Welcome Party. Please, send them to `magapeter@gmail.com`, or hand them in at the party. Participation in the research is conditional to a good result on these problems.

1. For a prime p and integers a, b, let

$$S(a,b;p) = \sum_{x=1}^{p-1} e\left(\frac{ax + b\bar{x}}{p}\right),$$

where $\bar{x}$ stands for the multiplicative inverse of x modulo p, and $e(z) = \exp(2\pi i z)$. Compute the value

$$\sum_{a=0}^{p-1} \sum_{b=0}^{p-1} S(a,b;p).$$

2. Assume p is a prime number, and χ is a nontrivial multiplicative character modulo p. Prove that for any integer $1 \leq N \leq p$,

$$\sum_{M=1}^{p} \left| \sum_{n=M+1}^{M+N} \chi(n) \right|^2 = N(p-N).$$

3. Let χ be a nontrivial multiplicative character modulo q, where $q > 1$. Prove that there exists a constant C depending only on q such that for any positive integer N,

$$\left| \sum_{n \geq \sqrt{N}} \frac{\chi(n)}{n} \right| \leq C N^{-1/2}.$$