POLINOMIALS

CLASSICAL ALGEBRA

The following set of exercises is usually a 4-5 weeks load.

1. Show that for $f(x) \in R[x]$ if $f(z) = 0$ for some $z \in C$, then $f(\overline{z}) = 0$, as well (with the same multiplicity).

2. Show that $z \in C$ is the root of $x^2 - 2Re(z) + |z|^2$, (this is a real polynomial.)

3. Show that over R every polynomial splits into quadratic and linear factors.

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4. Factor the following polynomials over C, R, Q:
$x^2 + x + 1, \qquad x^4 + 4, \qquad x^4 - 5x^2 + 6$

5. Factor the following polynomials over C, R:
$x^n - 1, \qquad x^n + 1, \qquad x^{2n} + x^n + 1$

6. Find the following gcd-s: $(x^n - 1, x^k - 1)$, $(x^n + 1, x^k + 1)$,

7. Show that $x^2 + x + 1 | x^{3m} + x^{3n+1} + x^{3k+2}$

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8. Find the sum of the squares, the sum of the cubes the product and the sum of the reciprocals of the (complex) roots of the polynomial $2x^4 + 2x + 3$.

9. Let $\alpha_1, \alpha_2, \alpha_3$ be the three roots of x^3+3x+1. Find the polynomials with roots $\alpha_1^2, \alpha_2^2, \alpha_3^2$ and $\alpha_1 + \alpha_2, \alpha_3 + \alpha_1, \alpha_2 + \alpha_3$

10. Let

$$x + y + z = a \text{ and } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a}$$

Show that one of x, y, z is equal to a.

11. Find the sum, the product, the sum of the squares of the n-th roots of unity.

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12. Find a polynomial of small degree s.t. $f(1) = 1$, $f(i) = i$, $f(-1) = -1$, $f(-i) = -i$,

13. Find a polynomial of small degree s.t. $f(1) = 1$, $f(i) = -i$, $f(-1) = -1$, $f(-i) = i$,

14. Find a polynomial of small degree s.t. $f(j) = 2^j$ for $j = 0, 2, \dots n$

15. Find the remainder of $x^{2000} + x^{1966} + x^{888} + x^{666}$ when it is divided by $x^2 - 1$ and $x^2 + 1$.

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16. Let $f(x)$ be a polynomial s.t. i is a 6 time root of f. Can the degree of f be 10 over R (over C)?

17. Find all polynomials s.t. $f'|f$.

18. For what b does the polynomial $x^n + bx^k + 1$ has a triple root?

19. Let f be a polynomial such that it has n nonzero coefficients. Show that the only possible root of f with multiplicity n is 0.

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20. Show that $3x^7 - 9x^5 + 6x^4 - 24x + 44$ is irreducible over Z. Hint: use (prove) the reversed Schoeneman-Eisenstien's criteria).

21. Prove that $5x^{13} - x + 6$ is irreducible over Z.

22. Is $3x^3 - 2x^2 + x - 10$ irreducible over Z?

23. Show an irreducible polynomial $p(x) \in Z[x]$ s.t. $p(\sqrt{2} + \sqrt{3}) = 0$.

24. Find all irreducible (monic) polynomials of degree 2 over F_5.

25. Is $x^3 - 4x^2 + x - 3$ $(x^3 - 4x^2 + x - 1)$ irreducible over F_7?

26. Using the above tricks prove that the product of 2 primitive polynomial is primitive.
