# Large Components in $r$-Edge-Colorings of $K_n$ Have Diameter at Most Five

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**Abstract:** Reflecting on problems posed by Gyárfás [Ramsey Theory Yesterday, Today and Tomorrow, Birkhäuser, Basel, 2010, pp. 77–96] and Mubayi [Electron J Combin 9 (2002), #R42], we show in this note that every $r$-edge-coloring of $K_n$ contains a monochromatic component of diameter at most five on at least $n/(r-1)$ vertices. © 2011 Wiley Periodicals, Inc. J Graph Theory 69: 337–340, 2012

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## 1. MONOCHROMATIC COMPONENTS IN EDGE-COLORINGS

The aim of Ramsey theory is to find large monochromatic structures in $r$-edge-colorings of a graph $G$. The most investigated case is when $G=K_n$; numerous articles and books have been published on this topic.

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A simple observation—in fact a remark of Erdős and Rado—is that every 2-edge-coloring of $K_n$ contains a monochromatic spanning component. One can also require further properties, e.g., every 2-edge-coloring of $K_n$ contains a monochromatic spanning component of diameter at most three [1, 7]. The maximum size of the largest monochromatic component of diameter at most two has been determined by Erdős and Fowler [2].

**Theorem 1** (Erdős and Fowler [2]). *Every 2-edge-coloring of $K_n$ contains a monochromatic component of diameter $\leq 2$ on at least $3n/4$ vertices.*

The following example shows that the bound given in Theorem 1 is sharp. Partition the set of vertices evenly into parts $A_1, A_2, A_3, A_4$ of size $\sim n/4$. For $j > i$ color all edges red between $A_i$ and $A_j$ if $j-i=1$, else color them blue. Color the edges inside each $A_i$ arbitrarily.

A *double star* is a tree obtained by connecting the centers of two vertex disjoint stars by an edge. Clearly, a double star has diameter three. In case of two colors, the maximum size of a monochromatic double star is similarly $\sim 3n/4$ [6]. A random coloring provides an example showing that the bound is best possible.

In case of three colors, the maximum size of a monochromatic component is $\sim n/2$. Moreover, Mubayi [7] showed that the diameter of the large monochromatic component is also bounded.

**Theorem 2** (Mubayi [7]). *Every 3-edge-coloring of $K_n$ contains a monochromatic component of diameter $\leq 4$ on at least $\lceil n/2 \rceil$ ($n/2+1$ if $n \equiv 2 \pmod 4$) vertices.*

In case of three colors, the maximum size of a monochromatic double star is not known yet; the best known lower and upper bounds are $4n/9$ and $n/2$ [6], respectively.

The aim of this article is to show that in every $r$-edge-coloring of $K_n$, there is large monochromatic component of a small constant diameter. The following theorem is a most general result of this type.

**Theorem 3** (Tonoyan [8]). *Let $D, r \geq 1$, $d \geq D$, $n \geq 2$. Then there exists an integer $t = R(D, r, n, d)$ such that every $r$-edge coloring of a graph $G$ on at least $t$ vertices with diameter $D$ possesses a monochromatic component of diameter at most $d$ on at least $n$ vertices.*

Notice that Tonoyan's general theorem gives no explicit bounds on the size and diameter of the monochromatic components. The size of the largest monochromatic component in $r$-edge-colorings of $K_n$ is related to the existence of affine planes of order $(r-1)$ as follows. Given positive integers $n, r$, let $f(n,r)$ be the largest number $t$ so that every $r$-edge-coloring of $K_n$ possesses a monochromatic component on at least $t$ vertices. The following theorem provides a lower bound on $f(n,r)$.

**Theorem 4** (Gyárfás [4]). *$f(n,r) \geq n/(r-1)$ and the equality holds if $(r-1)^2 | n$ and there is an affine plane of order $r-1$.*

See [5] for the recent developments and intriguing problems on monochromatic components in the case when $(r-1)^2$ does not divide $n$. Füredi [3] proved a stronger lower bound on $f(n,r)$ in the case when an affine plane of order $r-1$ does not exist.

**Theorem 5** (Füredi [3]). *If an affine plane of order* $r-1$ *does not exist, then* $f(n,r) \geq n/(r-1-(r-1)^{-1})$.

Theorems 4 and 5 suggest that it might be very difficult to determine $f(n,r)$ as these values depend on the existence of the affine plane of given order.

There are relatively large monochromatic components of diameter at most three.

**Theorem 6** (Mubayi [7]). *Every r-edge-coloring of* $K_n$ *contains a monochromatic component of diameter* $\leq 3$ *on at least* $n/(r-1+r^{-1})$ *vertices.*

This was improved in [6].

**Theorem 7** (Gyárfás and Sárközy [6]). *Every r-edge-coloring of* $K_n$ *contains a monochromatic double star on at least* $(n(r+1)+r-1)/r^2$ *vertices.*

We point out that Theorems 6 and 7 prove the existence of a smaller monochromatic component than Theorem 4. It may even be true that every $r$-edge-coloring of $K_n$ contains a monochromatic double star with at least $n/(r-1)$ vertices.

**Problem 1** (Gyárfás, Problem 4.2 in [5]). *For* $r \geq 3$, *is there a monochromatic double star of size asymptotic to* $n/(r-1)$ *in every r-coloring of* $K_n$?

A weaker version of the problem reads as follows.

**Problem 2** (Gyárfás, Problem 4.3 in [5]). *Given positive numbers n, r, is there a constant d (perhaps* $d=3$*) such that in every r-coloring of* $K_n$ *there is a monochromatic subgraph of diameter at most d with at least* $n/(r-1)$ *vertices*?

As we have already mentioned, for $r=2$ the affirmative answer to Problem 2 follows from [1, 7] with $d=3$ and for $r=3$ it follows from Theorem 2 with $d=4$.

## 2. BOUNDING THE DIAMETER

Although Mubayi [7] wrote that determining the size of the maximum monochromatic component of diameter $d \geq 3$ "seems to be difficult", we answer Problem 2 in the affirmative with $d=5$.

**Theorem 8.** *In every r-edge-coloring of* $K_n$, *there is a monochromatic subgraph of diameter at most* 5 *on at least* $n/(r-1)$ *vertices.*

***Proof.*** Consider an $r$-edge-coloring $c$ of $K_n$ on vertex set $[n]$ in colors $\{1,\ldots,r\}$. Let $G_i$ be the spanning subgraph of $K_n$ with the edges colored in $i$. If $G_i$ is not connected for some $1 \leq i \leq r$, $A$ being its component, then all edges in between $A$ and $[n]\setminus A$ are colored in the remaining $r-1$ colors. We shall use the following theorem, which is proved by a Cauchy–Schwarz-type counting argument.

**Theorem 9** (Mubayi [7]). *In every q-edge-coloring the complete bipartite graph* $K_{k,\ell}$, *there is a monochromatic double star of order* $(k+\ell)/q$.

Then, by Theorem 9 the coloring $c$ contains a required monochromatic subgraph even of diameter 3.

So, assume now that $G_i$ is connected for all $1 \le i \le r$. Let $v$ be a vertex of $K_n$. Then by $d_i(v)$ we denote the degree of $v$ in $G_i$. Further, assume wlog that the maximum monochromatic degree is attained at a vertex $v$ in the color $r$, i.e., $d_r(v) \ge d_i(w)$ for all $w \in [n]$ and $1 \le i \le r$. If all vertices in $G_r$ were within distance 2 from $v$, then $G_r$ would constitute a monochromatic component of diameter 4 on even $n$ vertices. So we are left with the case when there is a vertex $w$ at a distance 3 from $v$ in $G_r$. Hence, the neighbors of $v$ and $w$ are disjoint. We will consider two subcases. If $d_r(v)+d_r(w) \ge n/(r-1)-2$, then we are done. Indeed, the graph induced by the edges of a $v-w$ path of length 3 together with the edges incident to $v$ and $w$ constitutes a monochromatic subgraph of diameter 5 on at least $n/(r-1)$ vertices. Finally, let $d_r(v)+d_r(w) < n/(r-1)-2$. Clearly,

$$n-1 = (d_1(w)+\cdots+d_{r-2}(w))+(d_{r-1}(w)+d_r(w)) \tag{1}$$

$$\le (r-2)d_r(v)+(d_r(v)+d_r(w)) < (r-2)d_r(v)+n/(r-1)-2, \tag{2}$$

which is equivalent to

$$n\frac{r-2}{r-1}+1 < (r-2)d_r(v). \tag{3}$$

This implies $d_r(v) > n/(r-1)$, i.e., in this case we have a monochromatic star of the desired order.

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