# COURSE DESCRIPTION FOR "NUMERICS IN THE PRIME GEODESIC THEOREM"

## 1. The Topic

It is unclear what should be the correct order of magnitude of the remainder in the prime geodesic theorem for three dimensions. In two dimensions the numerics strongly suggest that in that case the error should be the square root of the main term, and one conjectures that this should be true in three dimensions as well. No numerics are available for three dimensions. The goal of the research class is to write a program to test the conjecture in this case.

For a short introduction to the topic see [1, Introduction]. For a discussion of the situation in three dimensions see [2, Introduction and Remark 3.1].

Prerequisites: basic of number theory, basic of analysis and complex analysis, basic of algebraic number theory (class groups of number fields), basic of programming.

## 2. Preliminary Assignment

Participation in the research is conditional to a good result on the following problems. Solutions should be sent to `cherubini.giacomo@renyi.mta.hu`.

• (1a) Let $M$ be a $2\times 2$ real matrix with determinant one, i.e. $M \in \mathrm{SL}(2,\mathbb{R})$. Prove that there exists a family of polynomials $p_k(t)$ such that

$$M^{k+1} = p_k(t)M + p_{k-1}(t)I,$$

where $t = \mathrm{tr}(M)$ and $I$ is the identity matrix.

(1b) Show that if $M$ has trace two, then $\mathrm{tr}(M^k) = 2$ for every $k$.

(1c) Show that if $M$ has trace strictly greater than two, then $\mathrm{tr}(M^k)$ is a strictly increasing function of $k$.

• (2) Consider the subgroup $\Gamma = \mathrm{SL}(2,\mathbb{Z}) < \mathrm{SL}(2,\mathbb{R})$, and let $M \in \Gamma$. In other words,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}, \quad ad-bc = 1.$$

Let $\mathbb{H} = \{x+iy, y>0\} \subseteq \mathbb{C}$. This is is called the hyperbolic plane. Consider the action of $\Gamma$ on $\mathbb{H}$ given by linear fractional transformation, i.e. for $M \in \Gamma$ and $z \in \mathbb{H}$ we set

$$Mz = \frac{az+b}{cz+d}.$$

(2a) Show that $Mz$ is again in $\mathbb{H}$. Moreover, show the following:

- If $|\mathrm{tr}(M)| < 2$ then $Mz = z$ occurs for some point in $\mathbb{H}$;
- If $|\mathrm{tr}(M)| = 2$ then $Mz = z$ occurs for some point in $\mathbb{Q}$ or for $z = \infty$;
- If $|\mathrm{tr}(M)| > 2$ then $Mz = z$ occurs for two distinct points of $\mathbb{R} \setminus \mathbb{Q}$.

(2b) Assume that $M \in \Gamma$ has trace strictly greater than two. Show that $M$ is conjugated in $\mathrm{SL}(2,\mathbb{R})$ to a diagonal matrix, i.e. $M = ADA^{-1}$, where $A \in \mathrm{SL}(2,\mathbb{R})$, and

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda \in \mathbb{R}, \quad \lambda > 1.$$

(2c) Consider the diagonal matrix $D$ as above. Compute the value

$$\ell(D) = \inf_{z \in \mathbb{H}} \int_z^{Mz} |dz| = \inf_{z \in \mathbb{H}} \int_{t_0}^{t_1} \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt,$$

where $z(t) = x(t) + iy(t)$, $t \in [t_0, t_1]$ is any curve joining $z$ and $Mz$. The number $\ell(D)$ is called the "*length*" of $D$.

- (3) Consider the functions

$$N(x) = \sum_{n \le x} d(n), \quad \text{and} \quad R(x) = \sum_{n \le x} r(n),$$

where $d(n)$ is the number of divisors of $n$ and $r(n)$ is the number of ways of writing $n$ as a sum of two squares. Compute the ratios

$$\frac{N(x)}{x \log x} \quad \text{and} \quad \frac{R(x)}{\pi x},$$

for $x = 10^j$, $j = 1, 2, 3, 4$.

## References

[1] K. Soundararajan and M. Young, *The Prime Geodesic Theorem*, https://arxiv.org/abs/1011.5486.

[2] O. Balkanova, D. Chatzakos, G. Cherubini, N. Laaksonen and D. Frolenkov, *Prime Geodesic Theorem in the 3-dimensional Hyperbolic Space*, https://arxiv.org/abs/1712.00880.