Geometry and mechanics of convex polyhedra

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Some background and preliminary questions

Polyhedra may be regarded as purely geometric objects, however, they are also often intuitively identified with solids. Among the most obvious sources of such intuition are dice which appear in various polyhedral shapes: while classical, cubic dice have 6 faces, a large diversity of other dice exists as well: dice with 2,3,4,6,8,10,12, 16, 20, 24, 30 and 100 faces appear in various games [9]. The key idea behind throwing dice is that each of the aforementioned faces is associated with a stable mechanical equilibrium point where dice may be at rest on a horizontal plane. A die is called *fair* if the probabilities to rest on any face (after a random throw) are equal [5], otherwise they are called *loaded* [4]. The concept of mechanical equilibrium may also be defined in purely geometric terms:

Definition 1 *Let P be a convex polyhedron, let* $\mathrm{inter}P$ *and* $\mathrm{bd}P$ *denote its interior and boundary, respectively and let $c \in \mathrm{inter}P$. We say that $q \in \mathrm{bd}P$ is an* equilibrium point *of P with respect to c if the plane through q and perpendicular to $[c, q]$ supports P at q.*

A *support plane* is a generalization of the tangent plane for non-smooth objects. While it is a central concept of convex geometry its name may be related to the mechanical concept of equilibrium. If c coincides with the center of mass of P, then equilibrium points gain intuitive interpretation as locations on bdP where P may be balanced if it is supported on a horizontal surface (identical to the support plane) without friction in the presence of uniform gravity. Equilibrium points may belong to three stability types: faces may carry stable equilibria, vertices may carry unstable equilibria and edges may carry saddle-type equilibria. Denoting their respective numbers by S, U, H, by the Poincaré-Hopf formula [1] for a convex polyhedron one obtains the following relation for them:

$$S + U - H = 2, \tag{1}$$

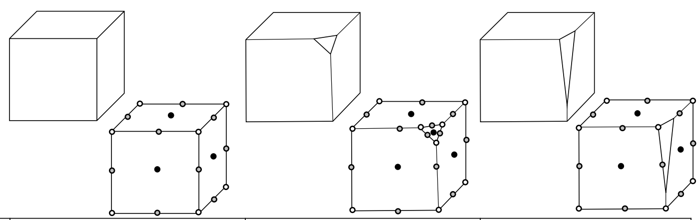
which is strongly reminiscent of the well-known Euler formula

$$f + v - e = 2, \tag{2}$$

relating the respective numbers f, v and e of the faces, vertices and edges of a convex polyhedron. In the case of a regular, homogeneous, cubic dice the formulae (1) and (2) appear to express the same fact, however, in case of irregular polyhedra the connection is much less apparent. While the striking similarity between (1) and (2) can only be fully explained via deep topological and analytic ideas [1], there exists some connection at an elementary, geometric level. To explain this connection, we define

$$\begin{aligned} N &= S + U + H, \\ n &= f + v + e. \end{aligned} \tag{3}$$

Figure 1 shows three polyhedra where the values for all these quantities can be compared.



f	6	7	7
v	8	10	10
e	12	15	15
n=f+v+e	26	32	32
S •	6	7	6
U ∘	8	10	8
H ∘	12	15	12
N=S+U+H	26	32	26
C=n-N	0	0	6

Figure 1: (a) Three polyhedra interpreted as homogeneous solids with given numbers for faces (f), vertices (v), edges (e), stable equilibria (S), unstable equilibria (U) and saddle-type equilibria (H), their respective sums $n = f+v+e$, $N = S + U + H$ and mechanical complexity $C = n - N$ (given in Definition 2). (b) Polyhedron in column $a3$ shown on the overlay of the (S, U) and (f, v) grids, complexity obtained from distance between corresponding diagonals.

The numbers S, U, H may serve, from the mechanical point of view, as a first-order characterization of P and via (1) the triplet (S, U, H) may be uniquely represented by the pair (S, U), which is called *primary equilibrium class* of P [8]. Based on this, we denote by $(S, U)^E$ the family of all convex polyhedra having S stable and U unstable equilibrium points with respect to their centers of mass. In an analogous manner, the numbers (v, e, f) (also called the f-vector of P) serve as a first-order combinatorial characterization of P, and via (2) they may be uniquely represented by the pair (f, v). Here, we call the the family of all convex polyhedra having v vertices and f faces the *primary combinatorial* *class* of P, and denote it by $(f,v)^C$. The face structure of a convex polyhedron P permits a finer combinatorial description of P. In the literature, the family of convex polyhedra having the same face lattice is called a combinatorial class; here we call it a *secondary combinatorial class*, for more details the interested reader is referred to [6]. While it is immediately clear that for any polyhedron P we have

$$f \geq S, v \geq U, \tag{4}$$

inverse type relationships (e.g. defining the minimal number of faces and vertices for given numbers of equilibria) are much less obvious.

Apparently, a necessary condition for any dice to be fair can be stated as $f = S$.

Question 1 *Construct polyhedra with $f = S$. (a) give examples for some values of f, (b) give a construction for arbitrary values of f.*

The opposite extreme case (when a polyhedron is stable only on one of its faces) appears to be far more complex and several papers [2, 3, 7] are devoted to this subject. Motivated by this intuition we define the *mechanical complexity* of polyhedra.

Definition 2 *Let P be a convex polyhedron and let $N(P), n(P)$ denote the total number of its equilibria and the total number of its* k-faces *(i. e., faces of k dimensions; now $k = 2, 1, 0$) respectively. Then $C(P) = n(P) - N(P)$ is called the mechanical complexity of P.*

Mechanical complexity may not only be associated with individual polyhedra but also with primary equilibrium classes.

Definition 3 *If $(S,U)^E$ is a primary equilibrium class, then the quantity*

$$C(S,U) = \inf\{C(P) : P \in (S,U)^E\}$$

is called the mechanical complexity of $(S,U)^E$.

Our goal is to find the values of $C(S,U)$ for all primary equilibrium classes.

Question 2 *Try to find $C(i,i)$ for $i = 1, 2, 3, 4, 5, 6$.*

Remark 1 *Recommended order: 4,5,6,3,2,1. The $i = 4$ case is trivial. On the other hand, if you find $C(1,1)$ then you might be eligible for the prize $P =$ 1.000.000$/C(1,1)$ in U.S. dollars.*

Question 3 *Try to find the nevessary and sufficient condition for $C(i,j) = 0$. You may guess and you may try to prove your guess.*

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