Covering by monochromatic components

Zoltán Király

1 Definitions

A graph $G = (V, E)$ with vertex set V and edge set E is complete if there is an edge between any pair of distinct vertices.

A graph is r-partite if there exists a partition of its vertex set $V = V_1 \cup \ldots \cup V_r$ into r classes, such that any edge has its endvertices in two different classes. A 2-partite graph is called bipartite.

An r-partite graph is complete if every pair of vertices from two different classes are connected by an edge.

A graph $G = (V, E)$ is called k-edge-colored if a coloring of its edges is also given, i.e., a function $c : E \to \{1, 2, \ldots, k\}$. Let $G_i = (V, E_i)$ where $E_i = \{e \in E : c(e) = i\}$ (the i-colored subgraph). The connected components of G_i are called monochromatic components of color i and a *monochromatic component* is a monochromatic component of color i for any $1 \le i \le k$.

A set of monochromatic components covers V if the union of their vertex sets equals V. Our central definitions are the following.

Suppose c is a k-edge-coloring of G, let $\mathrm{COV}(G, c)$ denote the minimum number of monochromatic components covering V. Moreover, let $\mathrm{COV}_k(G)$ denote $\max(\mathrm{COV}(G, c) : c$ is a k-edge-coloring of $G)$.

2 Warm-up exercises

Exercise 1. *Prove that if G is a complete graph, then* $\mathrm{COV}_2(G) = 1$.

Exercise 2. *Prove that if G is a complete graph, then* $\mathrm{COV}_k(G) \le k$.

Exercise 3. *Prove that if G is a complete bipartite graph, then* $\mathrm{COV}_2(G) \le 2$*; and this sharp whenever* $|V| > 2$.

Exercise 4. *Prove that if G is a complete bipartite graph, then* $\mathrm{COV}_k(G) \le 2k - 1$.

We call a k-edge-coloring *spanning* if for every $v \in V$ and for every $i \in \{1, \ldots, k\}$, there is an edge uv with $c(uv) = i$. Let $\mathrm{cov}_k(G)$ denote $\max(\mathrm{COV}(G,c) : c$ is a spanning k-edge-coloring of $G)$.

Exercise 5. *Prove that if G is a complete bipartite graph with at least two vertices in both classes, then $\mathrm{cov}_2(G) = 2$.*

3 Aim

Conjecture 1 (Ryser). *If G is a complete graph, then $\mathrm{COV}_k(G) \leq k - 1$.*

This is famous conjecture, strictly speaking it is a special case of Ryser's conjecture. However, it seems very hard to prove.

It would be very nice to prove this conjecture but we do not have too much hope on it.

Thus, our goal will be to study some special cases as well as some possible generalizations that were not studied before.

For example there is conjecture on complete bipartite graphs but not on complete r-partite graphs for $r > 2$.

Conjecture 2 (Gyárfás-Lehel). *If G is a complete bipartite graph, then $\mathrm{cov}_k(G) \leq 2k - 2$.*

Exercise 6. *Let G be a complete 3-partite graph. Give a lower and an upper bound on $\mathrm{cov}_k(G)$.*

Almost nothing is known for general graphs. For example a possible question can be the following.
Characterize those graphs G where $\mathrm{cov}_k(G) \leq k$.

Exercise 7. *Give a small 2-edge-colored graph where $\mathrm{COV}(G, c) = 3$.*