# Rigid and Globally Rigid Graphs

Research project

## Introduction

Let us start with a slightly informal description. More precise definitions will be given below. We shall work with $d$-dimensional bar-and-joint frameworks (also called geometric graphs) and analyse their rigidity properties. Such a framework consists of rigid bars (line segments) that meet at universal joints. The bars can move continuously in $d$-space so that the bar lengths and bar-joint incidences must be preserved. The framework is said to be *rigid* (in dimension $d$) if every such continuous motion of the framework results in a congruent framework, that is, the distance between each pair of joints is unchanged.

In the *graph* of the framework vertices correspond to the joints and edges correspond to the bars. It is known that if the framework is in sufficiently general position then rigidity depends only on its graph. For example, consider a framework consisting of four joints and five bars (so its graph consists of two triangles sharing an edge, see Figure 1). It is rigid in the plane but is not rigid in three-space. Note that if we remove an arbitrary bar, the framework is no longer rigid in the plane.

The framework is said to be *globally rigid* if every other framework with the same bars and same bar-joint incidences is congruent with the original one. Clearly, global rigidity implies rigidity. For example, a framework on the graph of Figure 1 is globally rigid on the line but is not globally rigid in two dimensions (consider the framework obtained by reflecting joint $u$ about the line of the bar which does not contain $v$).

The theory of rigid and globally rigid frameworks has surprisingly many applications in statics, structural analysis of molecules, sensor network localization, formation control, and elsewhere. There are lots of nice results as well as many open problems in this field.

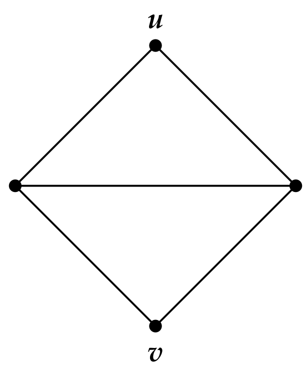


Figure 1: The graph of a framework with four joints and five bars.

## 1 Open problems

The open problems include (i) geometric, (ii) combinatorial, and (iii) algorithmic questions. Some examples are given below. We shall chose a problem suitable for the interested students.

We shall only consider two-dimensional frameworks and their graphs and (at least at the beginning) focus on the following open problems:

**Problem 1** *Characterize those graphs $G$ for which* `every` *bar-and-joint framework with underlying graph $G$ is rigid in the plane.*

The graph of Figure 1 is one of them. (Why?) It is known that such a graph $G = (V, E)$ satisfies $|E| \geq 2|V| - 3$, so one may start with the special case, when $G$ has exactly $2|V| - 3$ edges.

**Problem 2** *Given a graph $G$ and a cost function on its edges, design an algorithm which can find a minimum cost spanning subgraph of $G$ which is rigid (globally rigid).*

For the former version (concerning rigidity) there exists an efficient algorithm. For the latter version (global rigidity) there is no such algorithm and the problem is known to be NP-hard. Therefore we may look for good approximation algorithms.

**Problem 3** *Given an (approximation) algorithm for the previous problem, analyse its performance by implementing the algorithm and-or by searching for examples where the performance is worst possible.*

## Basic definitions

A $d$-dimensional *framework* is a pair $(G, p)$, where $G = (V, E)$ is a graph and $p$ is a map from $V$ to the $d$-dimensional Euclidean space $R^d$. We consider the framework to be a straight line *realization* of $G$ in $R^d$. Intuitively, we can think of a framework $(G, p)$ as a collection of bars and joints where each vertex $v$ of $G$ corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks $(G, p)$ and $(G, q)$ are *equivalent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs $u, v$ with $uv \in E$, where $dist(x, y)$ denotes the Euclidean distance between points $x$ and $y$ in $R^d$. Frameworks $(G, p)$, $(G, q)$ are *congruent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs $u, v$ with $u, v \in V$. This is the same as saying that $(G, q)$ can be obtained from $(G, p)$ by an isometry of $R^d$. We say that $(G, p)$ is *globally rigid* if every framework which is equivalent to $(G, p)$ is congruent to $(G, p)$.

A *motion* (or *flex*) of $(G, p)$ to $(G, q)$ is a collection of continuous functions $M_v : [0, 1] \to R^d$, one for each vertex $v \in V$, that satisfy

$$M_v(0) = p(v) \text{ and } M_v(1) = q(v)$$

for all $v \in V$, and

$$dist(M_u(t), M_v(t)) = dist(p(u), p(v))$$

for all edges $uv$ and for all $t \in [0, 1]$. The framework $(G, p)$ is *rigid* if every motion takes it to a congruent framework $(G, q)$.

For more details (and for the nice combinatorial results of this area) see the survey paper *Combinatorial rigidity: graphs and matroids in the theory of rigid frameworks* by Tibor Jordán, Technical report TR-2014-12, Egerváry Research Group, Budapest.

## Warm up exercises

Solve the first exercise and think about and make progress on some of the other warm-up exercises before starting the research project.

**Exercise 1.** Characterize the rigid bar-and-joint frameworks in $R^1$.

**Exercise 2.** Consider a bar-and-joint framework $(G, p)$ in $R^1$ and, to exclude degenerate situations, suppose that there is no algebraic relation between the coordinates of the joints (say, the set of the coordinates of the joints does not satisfy any non-zero polynomial with rational coefficients).

When is it globally rigid? The answer depends only on the graph $G$ of the framework. Try to find necessary and/or sufficient conditions.

**Exercise 3.** Consider a rectangular part of a square grid in the plane, say, with $n$ rows and $m$ columns of squares. Imagine that the points correspond to joints and the sides of the squares are all unit length bars. Such a framework is never rigid (why?). Try to make it rigid by adding a set of diagonal bars to the framework. (Diagonal bars are longer: each of them connects opposite corners of some square.) Characterize those sets of diagonal bars which make such a square grid framework rigid! What is the minimum number (in terms of $n$ and $m$) of diagonal bars you need to rigidify the framework?

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