Preliminary assignment for the research course "Hausdorff dimension of unions of lines"

BSM, 2017 Fall

At least one of these exercises should be done by the Welcome Party, at least two of them before our first meeting. If you send your solutions by e-mail to `tamas.keleti@gmail.com`, you can get feedback before we meet.

1. Find the definition of Hausdorff dimension in the internet, and directly from the definition prove that

   $$\dim(C \times [0,1]) \leq 1 + \frac{\log 2}{\log 3},$$

   where C denotes the Cantor set and dim denotes the Hausdorff dimension.

2. Let C be a subset of the plane and let E be the union of all lines of the form $y = ax + b$ such that $(a, b) \in C$. (In other words, E is the union of the lines "coded" by the "code-set" C.) For every angle φ let L_φ be the line through the origin in direction φ and let $\operatorname{proj}_\varphi C$ be the orthogonal projection of C to L_φ.

   Prove that if the (1-dimensional) Lebesgue measure of $\operatorname{proj}_\varphi C$ is zero for almost every φ then E has (2-dimensional) Lebesgue measure zero.

3. Let E be the union of some tangent lines of a fixed circle and let $D \subset [0, \pi)$ denote the set of directions of these lines. Prove that if D has positive Lebesgue measure then so has B.

Hint for Problems 2 and 3: Study the intersection of E with vertical lines and use Fubini theorem.

Have fun!