Let E be a subset of the unit square $[0, 1] \times [0, 1]$. Suppose that E is a gridlike set, so it is the union of some sub-squares of the form $[k/n, (k+1)/n] \times [l/n, (l+1)/n]$ for some fixed $n$.

a) Suppose that $E$ does not contain the 4 vertices of any axis-parallel rectangle of area at least 0.01. (In other words, whenever $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$ are all in $E$ then $(x_2 - x_1)(y_2 - y_1)$ is at most 0.01.) Give an upper estimate (as good as you can and of course smaller than 1) for the area of $E$. (Your estimate must be an absolute constant, like 0.23, it cannot depend on n, or on anything else.)

b) Suppose that $E$ has area 0.1. Find a positive c (as large as you can) for which you can prove that $E$ must contain the vertices of an axis-parallel rectangle of area at least $c$. (More precisely, prove that one can pick points of the form $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$ from $E$ so that $(x_2 - x_1)(y_2 - y_1)$ is at least $c$.)

c) Suppose that $E$ contains no rectangle of area at least $A$. (More precisely, whenever $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$ are all in $E$ then $(x_2 - x_1)(y_2 - y_1)$ is at most $A$.) Give an upper estimate (as good as you can) for the area of $E$. (The estimate depends on $A$, so it is a function of $A$.)

d) Suppose that $E$ has area $h$. Find a positive $c$ (as large as you can) for which you can prove that $E$ must contain a rectangle of area at least $c$. (More precisely, prove that one can pick points of the form $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$ from $E$ so that $(x_2 - x_1)(y_2 - y_1)$ is at least $c$.) Again, $c$ depends on $h$, so it is a function of $h$.

e) Prove that there exists a $C$ and an $h_0$ so that $c(h) = Ch^2/\log(1/h)$ is good for $h < h_0$.