Forbidden Configurations – Stability Theorems

A. Sali

sali.attila@renyi.hu

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We need some basic definitions. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Then an $m \times n$ simple matrix corresponds to a *simple hypergraph* or *set system* on m vertices with n edges. For a matrix A, let $|A|$ denote the number of columns in A. For a (0,1)-matrix F, we define that a (0,1)-matrix A has *no* F as a *configuration* if there is *no* submatrix of A which is a row and column permutation of F. Let $\mathrm{Avoid}(m,F)$ denote the set of all m-rowed simple matrices with no configuration F. Our main extremal problem is to compute

$$\mathrm{forb}(m,F) = \max_A\{|A| \ : \ A \in \mathrm{Avoid}(m,F)\}. \tag{1}$$

Let $\mathrm{Avoid}(m,\mathcal{F})$ denote the set of all m-rowed simple matrices with no configuration $F \in \mathcal{F}$. Define

$$\mathrm{forb}(m,\mathcal{F}) = \max_A\{|A| \ : \ A \in \mathrm{Avoid}(m,\mathcal{F})\}. \tag{2}$$

The following product is important. Let A and B be (0,1)-matrices. We define the product $A \times B$ by taking each column of A and putting it on top of every column of B. Hence if $|A| = a$ and $|B| = b$ then $|A \times B|$ is ab. Let I_m be the $m \times m$ identity matrix, I_m^c be the (0,1)-complement of I_m (all ones except for the diagonal) and let T_m be the triangular matrix, namely the (0,1)-matrix with a 1 in position i,j if and only if $i \le j$. The main conjecture states the following. Let $X(F)$ be the smallest p so that F is a configuration in $A_1 \times A_2 \times \ldots \times A_p$ for every choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$. Alternatively, assuming F is not a configuration in at least one of I, I^c , T , then $X(F)-1$ is the largest choice of p so that F is not a configuration in $A_1 \times A_2 \times \ldots \times A_p$ for some choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$.

Conjecture 0.1

$$\mathrm{forb}(m,F) = \Theta(m^{X(F)-1}) \tag{3}$$

A possible way to attack cases of the conjecture is to establish *stability results.* That is, statements like "if $A \in \mathrm{Avoid}(m,F)$ and $|A| > c_0 m^{X(F)-2}$, then A must contain a large

configuration of some of the products defining $X(F)$". One such example is a theorem of Anstee and Keevash from 2006
`https://www.sciencedirect.com/science/article/pii/S0195669806000771`.

In our proposed research we would look for small forbidden configurations that have a single product defining $X(F)$. One such example is

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

We know that $\text{forb}(m, F_1) = \lfloor \frac{m^2}{4} \rfloor + m + 1$, but we do not have yet a stability theorem for it. Stability theorems are interesting for their own sake, occur in several contexts in graph and hypergraph theory.

A good survey paper to get into the mood is Anstee's dynamic survey
`https://www.combinatorics.org/ojs/index.php/eljc/article/view/v8i1r4/pdf`
Introductory Problems:

1. Prove that $\text{forb}(m, F) = \text{forb}(m, F^c)$, where F^c is the $0-1$-complement of F.

2. What is $\text{forb}(m, I_2)$? What is $\text{forb}(m, \{I_2, T_2\})$?

3. Let F be a k-rowed matrix. Suppose we have $A \in \text{Avoid}(m, F)$ such that $|A| = \text{forb}(m, F)$. Consider deleting a row r. Let $C_r(A)$ be the matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. If we permute the rows of A so that r becomes the first row, then after some column permutations, A looks like this:

$$A = \begin{array}{c} r \\ \\ \end{array} \begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r(A) & & C_r(A) & C_r(A) & & D_r(A) \end{bmatrix}. \tag{5}$$

   where $B_r(A)$ are the columns that appear with a 0 on row r, but don't appear with a 1, and $D_r(A)$ are the columns that appear with a 1 but not a 0. Prove that

$$\text{forb}(m, F) \leq |C_r(A)| + \text{forb}(m-1, F). \tag{6}$$

4. Prove that

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \ldots + \binom{m}{0}. \tag{7}$$

5. Prove that

$$I_p \times T_p \in \text{Avoid}(m, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}). \tag{8}$$

6. Find the unique two-term product defining $X(F_1)$.