Preliminary assignment for the research course "Lipschitz and Hölder images of Cantor sets and more general self-similar sets and metric spaces"

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Solve as much as you can and send me your solutions by email. Don't hesitate to ask me if you need clarification or you have any question.

1. Find the definitions of Hausdorff and upper box dimension, and (directly from one of the equivalent definitions) prove that if X and Y are metric spaces and X can be mapped onto Y by a Lipschitz map then $\dim_{\mathrm{H}} Y \leq \dim_{\mathrm{H}} X$ and $\overline{\dim}_{\mathrm{B}} Y \leq \overline{\dim}_{\mathrm{B}} X$, where $\dim_{\mathrm{H}}$ and $\overline{\dim}_{\mathrm{B}}$ denotes the Hausdorff dimension and the upper box dimension, respectively.

2. A compact set $S \subset \mathbb{R}^n$ is called a self-similar set with the strong separation condition (SSC) if there exist contracting similarity maps $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}^n$ such that $S = \sqcup_{i=1}^k f_i(S)$, where $\sqcup$ denotes disjoint union. A metric space (Y, d) is called an ultrametric space if the triangle inequality is strengthened to $d(x,z) \leq \max\{d(x,y), d(y,z)\}$. Two metric spaces X and Y are called bilipshitz equivalent if there exists a bijection $f : X \to Y$ such that both f and its inverse are Lipschitz.

   Prove that in $\mathbb{R}^n$ any self-similar set with the SSC is bilipshitz equivalent to an ultrametric space.

3. For a metric space (X, d), $x_1, \ldots, x_n \in X$ and $s > 0$ define

$$Z^s(x_1, \ldots, x_n) = \max\left\{ \sum_{j=1}^{k-1} (d(x_{i_j}, x_{i_{j+1}}))^s : 1 = i_1 < \ldots < i_k = n \right\},$$

$$\delta^s(X) = \sup\left\{ \min_{\pi \in \mathrm{Sym}(n)} Z^s(x_{\pi(1)}, \ldots, x_{\pi(n)}) : \ x_1, \ldots, x_n \in X,\ n \geq 1 \right\},$$

   where by $\mathrm{Sym}(n)$ we mean the set of permutations of $\{1, \ldots, n\}$. Let X be a compact metric space and $\alpha > 0$. Prove that there exist a compact set $D \subset [0,1]$ and an α-Hölder onto map $g \colon D \to X$ if and only if $\delta^{1/\alpha}(X) < \infty$.

4. Let C be the classical Cantor set. Prove or disprove that a subset A of C can be mapped onto C by a Lipschitz map if and only if A contains a similar copy of C.

Have fun!