MPS demo sheet BSM

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This problem sheet contains selected problems from MPS. The first three topics have an easy a medium and a harder question, the last two a pair of an easier and a harder. If you see HPB at the end, it means that the problem is from the Hungarian Problem Book.

Number Theory:

N1. Is there a number n with digit sum 14 such that $2n$ has digit sum 15?

N2. Let $a_1^2+a_2^2+a_3^2+a_4^2+a_5^2=b^2$, where the variables are integers. Prove that not all of these numbers can be odd. (HPB 31.2)

N3. A number is called „double" if we repeat the digits of it (357357 is a double number). Prove that there are infinitely many perfect squares which are double numbers.

Algebra:

A1. There is a 100 stories high building. We have two identical glasses. We would like to know the highest floor from which it is possible to drop a glass and not have it broken on the ground. What is the least number of experimental drops we need in order to determine the answer?

A2. Let a, b, c and d be real numbers such that $a^2+b^2=c^2+d^2=1$ and $ac+bd=0$. Determine the value of $ab+cd$. (HPB 33.1.)

A3. Let a, b and c be pos. real numbers. Prove that $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}$.

When does equality occur?

Geometry:

G1. Prove that in any triangle, at most one side can be shorter than the altitude from the opposite vertex. (HPB 42.1.)

G2. Let A and B be two given points, l a given line. (a) Find the point P on l such that $AP+PB$ is a minimum. (b) Find the point P on l such that $|AP-PB|$ is a maximum.

G3. The hexagon $ABCDEF$ is inscribed in a circle. The sides AB, CD, EF are all equal in length to the radius. Prove that the midpoints of the other three sides determine an equilateral triangle. (HPB 41.3.)

Combinatorics and games:

C1. In a set of objects, each has one of two colors and one of two shapes. There is at least one object of each color and at least one object of each shape. Prove that there exist two objects in the set that are different both in color and in shape. (HPB 40.1.)

C2. Let F denote the number of 77-element subsets of $T=\{1, 2, \ldots, 2000, 2001\}$ for which the sum of the elements of the subset is even, and G the number of 77-element subsets for which the sum is odd. Which one is greater: F or G?

G1. We have 2019 coins on the table. Two players take some coins alternately. If there are n coins on the table we can take k such that $\gcd(n,k)=1$. The winner is who takes the last coin. How shall we play?

G2. We have $C=25$ coins on the table. Two players take some coins alternately. In the first step we may take 1. In the 2nd 1 or 2, …, in the n^{th} step at least 1, at most n. The winner is who takes the last coin. How shall we play?