Extremal and algorithmic problems in combinatorial rigidity

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BSM Research opportunities project

Introduction

A d-dimensional *bar-and-joint framework* (or simply *framework*) consists of rigid (fixed length) bars that meet at universal joints. The joints can move continuously in d-space so that the bar lengths and bar-joint incidences must be preserved. The framework is said to be *rigid* (in dimension d) if every such continuous motion results in a congruent framework, that is, the distance between each pair of joints is unchanged. The framework is said to be *globally rigid* if every other framework with the same graph and same bar lengths is congruent with it. Thus global rigidity implies rigidity.

Perhaps the first result in rigidity theory is due to A. Cauchy, from the 19th century. It states that a triangulated convex polyhedron (more precisely, the framework formed by its 1-skeleton) is rigid in three dimensions. It is well-known that the 1-skeleton of such a polyhedron is a maximal planar graph (also called a *triangulation*).

In the *graph* of the framework vertices correspond to the joints and edges correspond to the bars. We also say that the framework is a *realization* of its graph in $\mathbb{R}^d$. It is known that if the framework is in sufficiently general (called *generic*) position then (global) rigidity depends only on its graph (for every fixed d).

A *tensegrity framework* is a similar, but more general structure. It consist of bars, cables, and struts. The lengths of the bars are fixed, as above, but cables can get shorter and struts can get longer, providing only upper resp. lower bounds for the distance between their endpoints. The graph of the framework, in which edges are labeled as bars, cables, and struts, is called a *tensegrity graph*. We say that the bars, cables, and struts in a tensegrity graph or framework are its *members*. Rigidity and global rigidity of tensegrity frameworks can be defined similarly. In this more general setting these properties are no longer generic.

Combinatorial rigidity is concerned with the combinatorial and algorithmic properties of various families of graphs, defined by their rigidity properties. It is a very active field of research with plenty of nice results and lots of open problems.

Open problems

Rigidity theory is in the intersection of geometry, algebra, and combinatorics with several applications. It also includes quite a few questions concerning efficient algorithms. We shall chose some open problems suitable for the interested students. Three candidates are given below.

Problem 1 *Characterize the graphs $G = (V, E)$ for which every injective realization in the plane is rigid! (Injective means that no two vertices are coincident.)*

Although the following question is motivated by a result in rigidity theory, it is a question from algorithmic graph theory.

Problem 2 *Given a graph $G = (V, E)$, can we design an efficient algorithm that can decide whether G has a spanning subgraph $H = (V, F)$, $F \subseteq E$, which is a maximal outerplanar graph? (A graph is outerplanar if it has a planar embedding in which all vertices are on the boundary of the same face. It is maximal if adding any new edge destroys outerplanarity.)*

Tensegrity frameworks and graphs are substantially for difficult to analyse. It may still be possible to solve some extremal problems, like the following. A tensegrity graph is said to be *minimally* rigid (or globally rigid), if it is rigid (or globally rigid) in $\mathbb{R}^d$ but the removal of any member destroys this property.

Problem 3 *Determine the best possible upper bound on the number of members in a minimally globally rigid d-dimensional tensegrity framework, in terms of the number of its vertices! The cases $d = 1, 2$ are already challenging.*

Methods and prerequisites

We shall mostly use graph theoretic and combinatorial methods, so familiarity with the basics of graph theory is useful. Depending on the problems, in some cases geometric intuition is also useful, as well as familiarity with elementary linear algebra.

Basic definitions

A d-dimensional *framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to the d-dimensional Euclidean space $\mathbb{R}^d$. We consider the framework to be a straight line *realization* of G in $\mathbb{R}^d$. Intuitively, we can think of a framework (G, p) as a collection of bars and joints where each vertex v of G corresponds to a joint located at $p(v)$ and each edge to a rigid (that is, fixed length) bar joining its end-points. Two frameworks (G, p) and (G, q) are *equivalent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $uv \in E$, where $dist(x, y)$ denotes the Euclidean distance between points x and y in $\mathbb{R}^d$. Frameworks (G, p), (G, q) are *congruent* if $dist(p(u), p(v)) = dist(q(u), q(v))$ holds for all pairs u, v with $u, v \in V$. This is the same as saying that (G, q) can be obtained from (G, p) by an isometry of $\mathbb{R}^d$. We say that (G, p) is *globally rigid* if every framework which is equivalent to (G, p) is congruent to (G, p).

A *motion* (or *flex*) of (G, p) to (G, q) is a collection of continuous functions $M_v : [0, 1] \to \mathbb{R}^d$, one for each vertex $v \in V$, that satisfy

$$M_v(0) = p(v) \text{ and } M_v(1) = q(v)$$

for all $v \in V$, and

$$dist(M_u(t), M_v(t)) = dist(p(u), p(v))$$

for all edges uv and for all $t \in [0, 1]$. The framework (G, p) is *rigid* if every motion takes it to a congruent framework (G, q).

Warm up exercises

Solve at least four out of the next five exercises, and hand in the solutions, before starting the research project.

Exercise 1. Show that in the above mentioned result of Cauchy we cannot replace rigidity by global rigidity.

Exercise 2. For every $n \geq 3$ construct a minimally globally rigid tensegrity framework in $\mathbb{R}^1$ on n vertices, with exactly $2n-3$ members.

A vertex set S in a graph G is a *separator* if $G - S$ is disconnected. It is *minimal*, if no proper subset of S is a separator.

Exercise 3. A graph G is called *chordal* if every cycle in G of length at least four has a chord (that is, an edge which connects no non-consecituve vertices of the cycle). Let S be a minimal separator in G. Prove that the vertices in S are pairwise adjacent.

Exercise 4. Prove that the edge set of a graph G can be partitioned into (the edge sets of) two spanning trees of G if and only if G can be obtained from a single vertex by applying the following operations: (i) add a new vertex v to the graph and two new edges incident with v, (ii) delete an edge xy from the graph, and add a new vertex v and three new edges vx, vy, vz incident with v (thus the three new edges must include the edges from v to the endvertices of the deleted edge).

Exercise 5. Characterize the rigid frameworks in $\mathbb{R}^1$.

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