Title: Bond percolation via the belief propagation algorithm and spectra of Hashimoto matrices

Description: In former research courses we already studied spectra and clusters of graphs (together with the participating students we managed to publish two papers about this). Then we considered the spectra of the Laplacian and modularity matrices assigned to the graph and concluded that these spectra are mainly capable to find clusters of dense graphs. Recent results (e.g. [2]) show that for sparse graphs the spectrum of the so-called Hashimoto matrix is more capable for clustering purposes. The main objective of the proposed research is to study the spectral properties of this matrix and to extend the related algorithms for the sparse, several clusters case (with the techniques of [1]).

Belief propagation (also called message passing) is a recursion in a finite system, when in our case, at each step, a node asks its neighbors: “If I were not a member of the giant component, would you be a member there?”. Then the neighbors in turn ask their neighbors, and so on. After a back and forth run, the recursion terminates in finite steps.

Let G be a connected graph with n nodes and m edges, respectively. In case of bond percolation with edge-retention probability β, we have the random graph G^{β}, and concentrate on, when the giant component of it appears with increasing β. In [2], it is proved that the phase transition happens at the critical $\beta_C = \frac{1}{\lambda(\boldsymbol{B})}$, where $\lambda(\boldsymbol{B})$ is the largest eigenvalue of the $2m \times 2m$ *Hashimoto matrix* (also called *non-backtracking matrix*). It is not symmetric, so has complex eigenvalues too, but by the Frobenius theorem its largest modulus eigenvalue is positive real. The research problems to be studied are as follows.

- Characterize the spectrum of this matrix in case of sparse real life graphs. By the Ihara formula, the eigenvalues other than ± 1 are the same as those of a smaller $2n \times 2n$ matrix.
- Use the other structural (positive real) eigenvalues of $\boldsymbol{B}$ to find critical values of β when the second, third ... connected component of G^{β} appears.
- Use spectral clustering methods (by the eigenvectors corresponding to the structural eigenvalues) to find the clusters themselves.
- Apply the algorithms to real life graphs, where percolation corresponds to spreading an epidemic.

Prerequisites: Basic probability and graph theory, elementary matrix algebra, basics of complex numbers. (If a student is not familiar with one of these topics, he/she still can do the research with collaboration of other students having that expertise.) Best for students who intend to do research in spectral graph theory, bond percolation, and apply the theory for social networks.

Proposer: Prof. Marianna Bolla, DSc.

References:

[1] Bolla, M., Spectral Clustering and Biclusterig, Wiley (2013).

[2] Newman, M. E. J., Message passing methods on complex networks, https://arxiv.org/abs/2211.05054 (2022).