HELLY NUMBERS OF DISCRETE SETS

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Convexity plays a central role in geometry as well as in computer science. In this project, we will focus on combinatorial aspects of convexity, by studying discretized versions of Helly's theorem. Let us start with the central concepts.

Let C be a point set in $\mathbb{R}^d$. We say that C is *convex*, if along with any two points of it, C also contains the line segment between these two points. Equivalently, C is convex iff it is closed under taking finite *convex combinations*: for any $x_1, \dots, x_n \in C$, and any set of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for every i and $\sum_{i=1}^n \lambda_i = 1$, we have that

$$\sum_{i=1}^{n} \lambda_i x_i \in C\,.$$

Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be convex sets in $\mathbb{R}^d$. Helly's theorem, one of the cornerstones of combinatorial convexity, states the following:

Theorem 1. *If any $d+1$ sets of the family $\mathcal{C}$ share a common point, then there is a point common to all members of $\mathcal{C}$.*

The conclusion may also be written as $\cap\mathcal{C} \neq \emptyset$.

Helly's theorem has been generalized in many directions. There exist versions where convexity of the sets is not required, or where the ambient space differs from $\mathbb{R}^d$.

In the research project, we will be interested in the following problem. Let $\mathcal{S} \subset \mathbb{R}^d$ be a *discrete* set – that is, a set with no accumulation points (i.e. each point of the set has a neighborhood which contains no other points of $\mathcal{S}$). A typical example is that of the integer lattice $\mathbb{Z}^d$ consisting of all points with only integer coordinates. Now, we can ask for the *Helly number of $\mathcal{S}$* which is defined to be the smallest positive integer n for which the variant of Theorem 1 holds under the restriction that in the intersections, only points belonging to $\mathcal{S}$ are taken into account:

Definition 1.1. *For a discrete set $\mathcal{S} \subset \mathbb{R}^d$, let $H(\mathcal{S})$ denote the smallest positive integer such that for any family $\mathcal{C}$ of convex sets in $\mathbb{R}^d$ for which the intersection of any $H(\mathcal{S})$ or fewer members contain a point of $\mathcal{S}$ in common, then there exists a point of $\mathcal{S}$ common to all members of $\mathcal{C}$.*

The study of Helly numbers of discrete point sets have been very active recently. One of the important tools is due to Hoffmann:

Proposition 1.1. *If $\mathcal{S} \subset \mathbb{R}^d$ is a discrete set, then $H(\mathcal{S})$ equals the maximum number of vertices of an empty convex polytope in $\mathcal{S}$, that is, a convex polytope with vertices from $\mathcal{S}$ which does not contain any further point from $\mathcal{S}$.*

For $d = 2$, polytopes are replaced by polygons above.

The first introductory problem asks for applying this proposition to the case when $\mathcal{S}$ is the planar integer lattice.

Qualifying Problem 1. *Find the Helly number of the integer lattice* $\mathbb{Z}^d$.

First you should start by considering $d = 2$, then proceed to higher dimensions, at least to $d = 3$. It is easy to come up with an intuitively extremal empty convex polytope – can you prove that it is extremal indeed?

The next problem asks for a planar question. For a set $A \subset \mathbb{R}$, we will consider the usual direct product $A \times A = \{(a, b) : \ a, b \in A\}$.

Qualifying Problem 2. *Construct a sequence* $A = \{a_1, a_2, a_3, \ldots\}$ *of positive numbers for which the set* $A \times A \subset \mathbb{R}^2$ *has infinite Helly number.*

In the project, we will take $\mathcal{S}$ to be a variety of discrete sets in the plane and in higher dimensions. In particular, we will consider direct products of the form $S = A \times B$ with $A, B \subset \mathbb{R}$ being discrete sets. Another task will be to analyze 3-dimensional direct products. Variants of this problem have been studied by former BSM students.

Here is a final introductory problem, for which I do not expect a definite answer, rather just looking for some constructions.

Qualifying Problem 3. *Let* $A = \{2^n : n \in \mathbb{N}\}$, *and set* $\mathcal{S} = A \times A \times A \subset \mathbb{R}^3$. *Give a nontrivial lower bound on* $H(\mathcal{S})$ *which is as good as you can get.*

There are a number of interesting questions in this area, so an enjoyable – and hopefully productive – research project is guaranteed! Of course, we will spend the first half of the project learning all the necessary framework and methods.

Prerequisites: Calculus 1, some knowledge in geometry and combinatorics is preferred – however, we are going to cover all the material needed in the first part of the project.

If you are interested in participating the research project, please send your solutions to the above 3 Qualifying Problems to the email address ambruge@gmail.com.