Preliminary assignment for the online research course "Tiling a rectangle with squares using Diophantine approximation"

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Solve as much as you can and send me your solutions by email. If you cannot prove the claim of a problem, still you can use the result in a subsequent problem.

Don't hesitate to ask me if you need clarification or you have any question. If you send me some of your solutions or partial solutions early then you get early feedback.

1. Study simultanuous diophantine approximation in the literature, and using the methods you learn prove the following. For any real numbers $a_1, \dots, a_n$ there exists arbitrarily large positive integer q such that
$$\left\{ qa_i + \frac{1}{2} \right\} \in \left(\frac{1}{3}, \frac{2}{3} \right) \quad (i = 1, \dots, n),$$
where $\{.\}$ denotes the fractional part.

2. Let $S_1, \dots, S_n$ be axis-parallel squares in the plane. Prove that there exist a translation vector $v \in \mathbb{R}^2$ and a scaling factor $b > 0$ such that every square $bS_i + v$ has only noninteger coordinates and intersects the same number of vertical and horizontal lines of the lattice $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z})$ (that is, the same number of lines of the form $\{k\} \times \mathbb{R}$ $(k \in \mathbb{Z})$ and lines of the form $\mathbb{R} \times \{k\}$ $(k \in \mathbb{Z})$) and this number is nonzero at least for one i.

3. (Max Dehn's theorem) Prove that a rectangle can be tiled with finitely many squares (of possibly different sizes) if and only if the ratio of the sides of the rectangle is rational.

4. a) A rectangle is tiled with n squares (of possibly different sizes). Show that the ratio of the sides of the rectangle can be written as p/q, where p and q are integers and $p, q \leq C^n$ for some absolute constant C.

   b) For every positive integer n find a rectangle that can be tiled with n squares and the sides of the rectangle are coprime integers and both are greater than c^n for some absolute constant $c > 1$.

Have fun!