Forbidden Configurations Research Proposal

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We need some basic definitions. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Then an $m \times n$ simple matrix corresponds to a *simple hypergraph* or *set system* on m vertices with n edges as columns of the matrix are the characteristic vectors of sets in the set system. For a matrix A, let $|A|$ denote the number of columns in A. For a (0,1)-matrix F, we define that a (0,1)-matrix A has F as a *configuration* if there is a submatrix of A which is a row and/or column permutation of F, in notation $F \prec A$. Let $\text{Avoid}(m, F)$ denote the set of all m-rowed simple matrices with no configuration F. The fundamental extremal problem is to compute

$$\text{forb}(m, F) = \max_{A}\{|A| \colon A \in \text{Avoid}(m, F)\}. \tag{1}$$

Let $\text{Avoid}(m, \mathcal{F})$ denote the set of all m-rowed simple matrices with no configuration $F \in \mathcal{F}$. Define

$$\text{forb}(m, \mathcal{F}) = \max_{A}\{|A| \;:\; A \in \text{Avoid}(m, \mathcal{F})\}. \tag{2}$$

The following product is important. Let A and B be (0,1)-matrices. We define the product $A \times B$ by taking each column of A and putting it on top of every column of B. Hence if $|A| = a$ and $|B| = b$ then $|A \times B|$ is ab. For example, the vertex-edge incidence matrix of the complete bipartite graph $K_{m/2,m/2}$ is $I_{m/2} \times I_{m/2}$. Let I_m be the $m \times m$ identity matrix, I_m^c be the (0,1)-complement of I_m (all ones except for the diagonal) and let T_m be the triangular matrix, namely the (0,1)-matrix with a 1 in position i, j if and only if $i \leq j$. The following is main motivating conjecture.

Conjecture 0.1 *[2] Let F be a $k \times \ell$ matrix with $F \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $X(F)$ denote the largest p such that there are choices $A_1, A_2, \ldots, A_p \in \{I_{m/p}, I^c_{m/p}, T_{m/p}\}$ so that $F \not\prec A_1 \times A_2 \times \cdots \times A_p$. Then* $\text{forb}(m, F) = \Theta(m^{X(F)})$.

Many special cases have been verified, for details one may consult [1]. In the current research proposal we extend the questions from (0,1)-matrices to r-matrices. An r-matrix is a matrix with entries from $\{0, 1, \ldots, r-1\}$. Concepts of simple matrix, configuration extend naturally and Avoid$(m, r, \mathcal{F})$ denote the set of all m-rowed simple r-matrices with no configuration $F \in \mathcal{F}$ similarly let forb$(m, r, \mathcal{F}) = \max_A\{|A| : A \in \text{Avoid}(m, r, \mathcal{F})\}$. It was proven in [3] that forb$(m, r, \mathcal{F})$ is of polynomial order of magnitude if and only if $\mathcal{F}$ contains an (i, j)-matrix for every $0 \le i < j \le r-1$, where an (i, j)-matrix is a matrix with entries i and j. This gives two major research directions. First, we may look for special collections of (i, j)-matrices $\mathcal{F}$. Let F be a $(0, 1)$-matrix, define $F(i, j)$ as the (i, j)-matrix given by replacing the 0's in F with i's and the 1's in F with j's. Furthermore, let

$$\text{Sym}(F) = \{F(i, j) : 0 \le i < j \le r - 1\}.$$

A similar set is

$$\mathbf{S}(F) = \{F(i, j) : 0 \le i, j \le r - 1, i \ne j\}.$$

Investigations of these have been started but there are many open problems. One interesting problem is the following. If F^c denotes the 0-1-complement of a (0,1)-matrix F, then we see that $\mathbf{S}(F) = \text{Sym}(F) \cup \text{Sym}(F^c)$, in particular forb$(m, r, \text{Sym}(F)) =$ forb$(m, r, \mathbf{S}(F))$ if $F = F^c$. One should find F where forb$(m, r, \text{Sym}(F))$ and forb$(m, r, \mathbf{S}(F))$ are of different order of magnitude.

Another possible direction is not to worry about exponential bounds. One may look at forb(m, r, F) for a (0,1)-matrix F and try to find exact exponential bounds. Some preliminary investigations have been done in this direction, as well. Let K_k denote the $k \times 2^k$ (0,1)-matrix of all distinct columns.It follows from an early result of Alon that forb(m, r, K_2) is the number of columns with at most one entry 1. We conjecture that for $F = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$, forb$(m, r, F)$ is also the number of columns with at most one entry 1.

Here are some practice problems to get into the mood:

1. Prove that forb$(m, F) =$ forb(m, F^c), where F^c is the $0-1$-complement of F.

2. What is forb(m, I_2)? What is forb$(m, \{I_2, T_2\})$?

3. Let F be a k-rowed matrix. Suppose we have $A \in \text{Avoid}(m, F)$ such that $|A| =$ forb(m, F). Consider deleting a row r. Let $C_r(A)$ be the matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. If we permute the rows of A so that r becomes the first row, then after some column permutations, A looks like this:

$$A = \begin{matrix} r \end{matrix}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r(A) & & C_r(A) & C_r(A) & & D_r(A) \end{bmatrix}. \tag{3}$$

   where $B_r(A)$ are the columns that appear with a 0 on row r, but don't appear with a 1, and $D_r(A)$ are the columns that appear with a 1 but not a 0. Prove that

$$\text{forb}(m, F) \le |C_r(A)| + \text{forb}(m - 1, F). \tag{4}$$

4. Let K_k denote the $k \times 2^k$ simple $0-1$-matrix (configuration). Use the decomposition (3) and the inequality (4) to prove that $\mathrm{forb}(m, K_k) = O(m^{k-1})$.

5. Prove that
$$\mathrm{forb}(m, K_k) \geq \binom{m}{k-1} + \binom{m}{k-2} + \ldots + \binom{m}{0}. \tag{5}$$

6. Do we have equality in (5)?

7. Prove that
$$I_p \times T_p \in \mathrm{Avoid}(m, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}). \tag{6}$$

8. * Assume that we consider forbidden configurations of $\{0,1,2\}$-matrices. Let $\mathcal{T}_{i,j,k} = \left\{ \begin{pmatrix} j & k \\ i & j \end{pmatrix} \text{ for } i,j,k \in \{0,1,2\} \right\}$. Here we assume that $i = j = k$ does not hold. What is $\mathrm{forb}(m, 3, \mathcal{T}_{i,j,k})$?

References

[1] R.P. Anstee. A Survey of forbidden configurations results. *Elec. J. of Combinatorics* 20, DS20, (2013).

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[3] Z. Füredi and A. Sali, Optimal multivalued shattering. *SIAM Journal on Discrete Mathematics*, **26**(2) 737-744, 2012.