THE LAYER NUMBER OF HIGH-DIMENSIONAL REGULAR GRIDS

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This project is the continuation of a previous research project at BSM [1], in which we studied layer numbers of certain finite point sets.

In order to reach the problem, we should start off with some definitions. Let C be a point set in $\mathbb{R}^d$. We say that C is *convex*, if along with any two points of it, C also contains the line segment between these two points. Equivalently, C is convex iff it is closed under taking finite *convex combinations*: for any $x_1, \dots, x_n \in C$, and any set of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_i \geq 0$ for every i and $\sum_{i=1}^n \lambda_i = 1$, we have that

$$\sum_{i=1}^n \lambda_i x_i \in C \,.$$

Let X be a finite set of points in $\mathbb{R}^d$. The *convex hull* of X is the smallest convex set containing X. We may imagine this as taking a large balloon containing all the points of X. If we pierce the balloon, it is going to deflate, being stopped by only the points of X. The convex hull of X is then this "deflated" balloon.

If X is finite, its convex hull will be in fact a polytope with some finite number of vertices – these are points of X, called the *extreme points* of X. (Polytopes are higher dimensional analogues of convex polygons in the plane). Using the above example, the extreme points of X are the ones that will suspend our deflated balloon.

In the current setting, the set of extreme points of X is referred to as the *convex layer* of X, in notation, $V(X)$. We can also call it the *first* convex layer.

Now, imagine that we drop the points of $V(X)$ from X, and call the remaining set X_1. For convenience, let $X_0 = X$. Clearly, $X_1 \subset X_0$.

Why stop here? We may replace X_0 by X_1, and repeat the process: take the extreme points of X_1 (that is, the convex layer), and delete them from X_1. The remaining set shall be called X_2. And so on! Thus, we may define recursively the sets X_i as

$$X_i = X_{i-1} \setminus V(X_{i-1})$$

for $i = 1, 2, \dots$. Clearly, this is a nested sequence: $X_i \subset X_{i-1}$. It is also easy to see, that after finitely many steps, we arrive at the empty set (why?). This process is called the *peeling process*.

We are interested in the number of steps the peeling process takes to terminate (i.e. to reach the empty set), given the set X. This number is called the *layer number* of X, denoted as $L(X)$. We want to give *asymptotic estimates* for $L(X)$ in terms of the number of points in X, that we are going to denote by n.

Introductory Problem 1. Give an upper bound on $L(X)$ that is valid for every set X consisting of n points. Try to give the best possible bound, and show that it is tight. Does this bound depend on the dimension of the ambient space? If yes, what is its relation to d? If not, how could we put an extra condition on X that somehow forces the dimension to play a role?

Of course, the layer number is related intimately to the structure of X. We expect different behaviours for different sets.

In our research project [1], we determined almost sharp upper and lower bounds for the layer number of finite point sets contained in the unit ball which are fairly equally distributed. Our example for the extremal sets is a recursive construction using the already established, lower dimensional sets as building blocks.

In the current research project, we will take a different direction. Our goal will be to determine the layer number of d-dimensional sets which are, in some sense, as evenly distributed as possible – we are interested in the *regular grid* $G_n^d = [n]^d$. That is, G_n^d consists of all the points in $\mathbb{R}^d$ all of whose coordinates are in the set $\{1, 2, \ldots, n\}$. (The planar case is familiar to all of you – just take a checkerboard!) It is easy to see that the number of points in G_n^d is n^d.

Introductory Problem 2. Calculate the layer number of $[k]^2$ for $k = 1, 2, \ldots, 6$. What do you think, how do convex layers of $[k]^2$ look like for large values of k?

It was proven in [3] that the layer number of G_n^2 is proportional to $n^{4/3}$ (for the ones familiar with computer science, its order of magnitude is $\Theta(n^{4/3})$). The main goal of the research will be trying to extend this result to higher dimensions:

Open research question 1. Determine the order of magnitude of the layer number of the d-dimensional regular grid G_n^d.

On the other hand, it was shown in [3] that in the plane it is possible to construct non-uniform grids with nearly as many convex layers as the number of points. That is, the construction shows that there exists a non-uniform grid with N points whose layer number is at least cN, with some positive constant independent from N. Our next question is whether such a construction also exist in higher dimensions.

Open research question 2. Is it possible to construct a non-uniform grid G^d in $\mathbb{R}^d$ with its layer number satisfying

$$L(G^d) > cN$$

with some positive constant c?

In computer science terminology, the above condition is written as $L(X) = \Omega(N)$.

Finally, if time permits, we may study random point sets: here, we select n random, uniform points in the unit disk – this serves as our set X. It was proven in [2] that if the points are taken according to the uniform distribution, then $L(X) = \Theta(N^{2/3})$) where $N = |X|$. Thus, the layer number of a uniform random points set has the same order of magnitude as the case of the regular grid. We may study this problem when the random points are not taken according to the uniform distribution, but with respect to the *normal distribution* on the plane.

Prerequisites: some geometry and combinatory knowledge would be preferred, but we will cover the topics needed.

If you are interested in participating the research project, please send your solutions to the above two introductory problems no later than May 28 to the email address ambruge@gmail.com.

References

[1] G. Ambrus, P. Nielsen, and C. Wilson, *The layer number of evenly distributed sets in high dimensions.* In preparation.

[2] Ketan Dalal, *Counting the onion.* Random Structures Algorithms **24** (2004), no. 2., 155–165.

[3] Sariel Har-Peled and Bernard Lidický, *Peeling the grid.* SIAM J. Disc. Math. **27**(2013), no. 2., 650–655.

[4] Ilkyoo Choi, Weonyoung Joo, and Minki Kim, *The layer number of α-evenly distributed point sets.* Manuscript, 2019.

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