1 Knots and knot invariants

1.1 Introduction

Definition 1. *A* knot K *is a* $\mathcal{C}^\infty$ *function* $K : S^1 \to S^3 (= \mathbb{R}^3 \cup \{\infty\})$*, where* $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$*.* K *has three* $\mathcal{C}^\infty$ *coordinate functions* $(x(\vartheta), y(\vartheta), z(\vartheta))$ *giving embeddings, that is,* K *is an embedding of the circle into the 3-sphere/* S^1 *into* S^3*.*

Definition 2. *Two knots* K_0 *and* K_1 *are* isotopic *if there is a smooth map* $K : S^1 \times [0, 1] \to \mathbb{R}^3 \times [0, 1]$*, such that*

- $K_t = K |_{S^1 \times \{t\}}$ *is a knot,*
- *for* $t = 0$ *and* 1*,* K_0 *and* K_1 *are the given knots.*

This means that K_0 *can be moved into* K_1 *without cutting.*

The main problem of knot theory is to distinguish knots from each other. There are several ways to study them: the most topological one is to view knots as subspaces of $\mathbb{R}^3$, while a rather combinatorial idea is to consider their projection to a plane. For a generic choice of this plane, we can assume that the projection of the knot has at most double points. Moreover, it is an immersion with finitely many double points. At the double points, we illustrate the strand passing under as an interrupted curve segment, see Figure 1. The resulting diagram D is called the *knot diagram* of K, and the neighborhood of a double point a *crossing*. It is obvious that the knot diagram determines the knot up to isotopy.

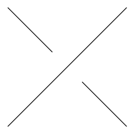

Figure 1: Crossing in a diagram

The trivial knot is usually called the unknot and is denoted by U. Figure 2 shows two diagrams of U.

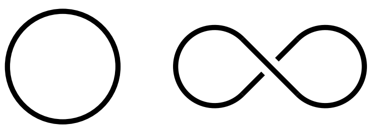

Figure 2: Two diagrams of the unknot

Definition 3. *An* l-component link *is the disjoint union of* l *knots.*

There are three important local modifications of a knot diagram, called the Reidemeister moves, shown on Figure 3:

- R_1: Twisting or untwisting a strand,
- R_2: Moving a loop over another strand or removing it from that,
- R_3: Sliding a string over or under a crossing

The other parts of the diagram stay unchanged. It is easy to see that the Reidemeister moves preserve the knot up to isotopy, but also more is true:

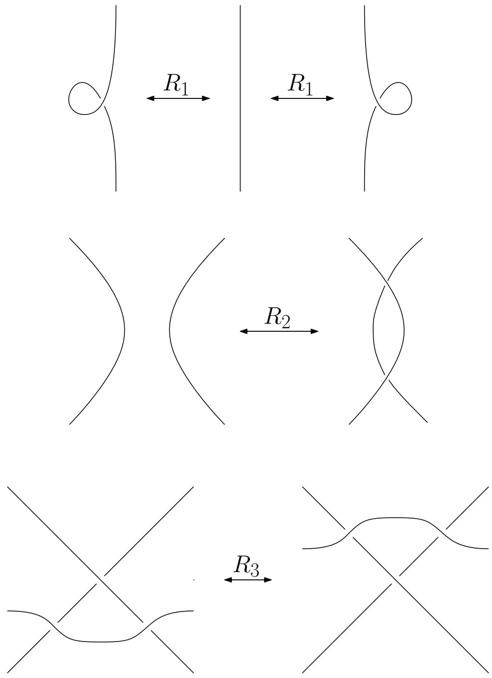

Figure 3: Reidemeister moves

Theorem 4 (Reidemeister)**.** *Two knot diagrams D_1 and D_2 represent equivalent knots if and only if they can be transformed to each other by a finite sequence of Reidemeister moves and planar isotopies.*

To classify knots, is very useful to introduce quantities that stay unchanged under isotopy. These are called *knot invariants*. In this series of lectures we observe how knot theory and knot invariants has evolved in the past years. Our main goal is to measure the complexity of knots. To this, we focus on the following questions:

1. How to identify the unknot $\mathcal{U}$? Can we give an invariant that detects $\mathcal{U}$? (We are going to see such an example.)
2. If a knot is not the trivial one, how far is it from being the unknot? To determine this, we introduce three numbers:
   (a) The unknotting number $u(K)$
   $u(K)$ is the minimal number of crossing changes needed to transform K to U. It can be shown as an exercise, that $u(K)$ always exists.
   (b) The crossing number $cr(K)$
   $cr(K)$ is the minimal number of crossings K has in any diagram. The crossing number is always a non-negative integer, and equals 0 if and only if K is the unknot.

   Remark 5. *There is no non-trivial knot with 1 or 2 crossings.*

   These provide upper bounds on the complexity of knots. However, to obtain more precise results, for example to know the minimal values of the above numbers, we need lower estimates too. Now we show some ways to get lower bounds.

1.2 Three-colorings

Definition 6. *A diagram D is* three-colorable *if we can associate a color out of {red, white, green} to every arc of D such that*

- *we use at least two colors,*
- *at each crossing either only one or all the three colors are used to color the meeting arcs.*

Example 7. *The following diagrams of the unknot and the figure-8 knot are not three-colorable, while the ones for the left-handed and the right-handed trefoil are.*

It is obvious that this property depends on the chosen diagram of the knot. But how do we know in case of a not three-colorable diagram if there exists another choice of the projection that is three-colorable?

Theorem 8. *If D_1 and D_2 are two diagrams corresponding to the same knot, then D_1 is three-colorable if and only if D_2 is three-colorable.*

Proof. Isotopies obviously do not change this property of a diagram. Therefore it is enough to check that nor do Reidemeister moves, see Figure 4. □

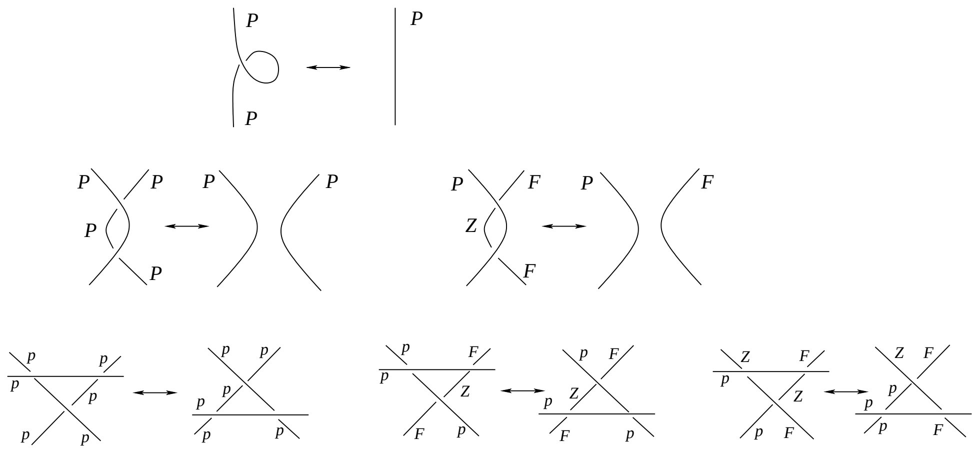

Figure 4: Invariance of three-colorability under Reidemeister moves

As a consequence, we see that there are at least two different knots: the unknot is not three-colorable, while the trefoil knot is.

We can improve this idea by using more colors, as follows:

Definition 9. *Let q be an odd prime. A diagram D is q-*colorable *if the arcs of D can be colored by $\{0, ..., q-1\}$ such that*

- *we use at least two colors,*
- *at each crossing colored with a, b and c, $a + c \equiv 2b \pmod{q}$.*

Example 10. *The trefoil knot is not 5-colorable, but the figure-8 knot is.*

Example 11. *For p and q odd primes the diagram with p twists in Figure 5 admits a q-coloring if and only if $p = q$.*

We start coloring the arcs with colors a and b according to the figure. Then, at the first crossing we have to use color $2b - a$ to satisfy $a + c \equiv 2b \pmod{q}$. At the next crossing we can only choose color $3b - 2a$, etc. After reaching the last crossing, we get that the first colored arcs have to be of the colors $pb - (p-1)a$ and $(p+1)b - p$. These give us two equations:

$$pb - (p-1)a \equiv a \pmod{q}$$

$$(p+1)b - p \equiv b \pmod{q}.$$

We get $p(b-a) \equiv 0 \pmod{q}$, that is, $q|p(b-a)$. This means that either $q|b-a$, meaning that all the arcs were of the same color, or $q|p$, meaning that $p = q$.

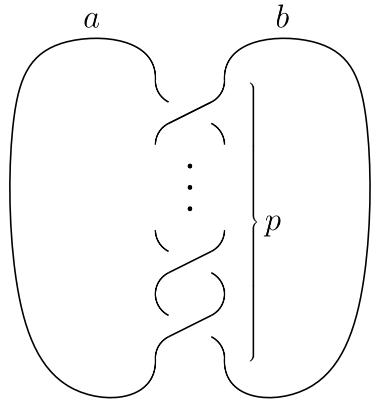

Figure 5: This diagram admits a q-coloring if and only if $p = q$.

We can generalize the idea of q-colorings by allowing to use only one color. Let $\mathcal{C}(q, K)$ denote the set of all generalized q-colorings.

Theorem 12.

- *$\mathcal{C}(q, K)$ forms a vector space over the finite field $\mathbb{F}_q = \{0, ..., q-1\}$.*
- *$\mathcal{C}(q, K)$ is independent of the choice of D, so it is a knot invariant.*
- *There exists a q-coloring if and only if* $\dim \mathcal{C}(q, K) > 1$.
- $\dim \mathcal{C}(q, K_1 \# K_2) = \dim \mathcal{C}(q, K_1) + \dim \mathcal{C}(q, K_2) - 1$
- *For the trefoil knot T and the connected sum operation $\#$,*

  $$nT = \underbrace{T \# T \# \cdots \# T}_{n}$$

  has $\dim \mathcal{C}(3, nT) = n + 1$ *- different for every value of n. Therefore, the knots obtained this way are all different.*

Theorem 13. *For the unknotting number $u(K) \geq$* $\dim \mathcal{C}(3, K) - 1$.

The idea of the proof is that a crossing change can change $\mathcal{C}(3, K)$ by at most 1 dimension. As a corollary, we get that nT has $u(nT) \geq n$.