Preliminary assignment for the online research course "Dimension of union of polygons"

BSM, 2020 Summer
Tamás Keleti
tamas.keleti@gmail.com

Solve as much as you can and send me your solutions by email. Don't hesitate to ask me if you need clarification or you have any question. If you send me some of your solutions or partial solutions early then you get early feedback.

Find equivalent definitions of upper box dimension (also called upper Minkowski dimension and denoted by $\overline{\dim}_B$) and packing dimension (also called upper packing dimension or modified upper box dimension, denoted by $\dim_P$, $\overline{\dim}_P$ or $\overline{\dim}_M B$), and try to solve these exercises by using always the most convenient definition.

1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map, let $E \subset \mathbb{R}^n$ and let $f(E)$ denote the image of the set E. Prove that

   a) $\overline{\dim}_B(f(E)) \le \overline{\dim}_B(E)$, provided E is bounded, and

   b) $\dim_P(f(E)) \le \dim_P(E)$ for any E.

2. Prove some of the following formulae for bounded subsets $A, B \subset \mathbb{R}$:

$$\begin{aligned}
\overline{\dim}_B([0,1]) &= 1,\\
\dim_P([0,1]) = \dim_P(\mathbb{R}) &= 1,\\
\overline{\dim}_B(A \times [0,1]) &= 1 + \overline{\dim}_B(A),\\
\dim_P(A \times \mathbb{R}) = \dim_P(A \times [0,1]) &= 1 + \dim_P(A),\\
\overline{\dim}_B(A \times B) &\le \overline{\dim}_B(A) + \overline{\dim}_B(B),\\
\dim_P(A \times B) &\le \dim_P(A) + \dim_P(B).
\end{aligned}$$

3. Let $B \subset \mathbb{R}^2$. Suppose that for every $p \in [0,1] \times [0,1]$ the set B contains the boundary of an equilateral triangle T such that the center of T is p and one of the edges of T is parallel to the x-axis. Prove that this implies that $\dim_P(B) \ge 5/3$. (You can use any of the above statements, even if you have not proved them.)

   HINT: You can use the argument sketched in the last paragraph of our paper "Squares and their centers".

Have fun!