Forbidden Configurations introductory problems

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We need some basic definitions. Define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. Then an $m \times n$ simple matrix corresponds to a *simple hypergraph* or *set system* on m vertices with n edges. For a matrix A, let $|A|$ denote the number of columns in A. For a (0,1)-matrix F, we define that a (0,1)-matrix A has *no* F as a *configuration* if there is *no* submatrix of A which is a row and column permutation of F. Let $\text{Avoid}(m, F)$ denote the set of all m-rowed simple matrices with no configuration F. Our main extremal problem is to compute

$$\text{forb}(m, F) = \max_A \{|A| \ : \ A \in \text{Avoid}(m, F)\}. \tag{1}$$

The following product is important. Let A and B be (0,1)-matrices. We define the product $A \times B$ by taking each column of A and putting it on top of every column of B. Hence if $|A| = a$ and $|B| = b$ then $|A \times B|$ is ab. Let I_m be the $m \times m$ identity matrix, I_m^c be the (0,1)-complement of I_m (all ones except for the diagonal) and let T_m be the triangular matrix, namely the (0,1)-matrix with a 1 in position i, j if and only if $i \leq j$. Problems:

1. Prove that $\text{forb}(m, F) = \text{forb}(m, F^c)$, where F^c is the $0-1$-complement of F.
2. Let $A \in \text{Avoid}(m, I_2)$ be a simple matrix with no repeated rows. Define a directed graph $G = (V, E)$, where V is the set of rows of A and (i, j) is an arc in E if there is no column of A with a 0 in row i and and a 1 in row j. Show that G is a tournament (directed complete graph). Use this to prove that $\text{forb}(m.I_2) = m+1$.
3. Let F be a k-rowed matrix. Suppose we have $A \in \text{Avoid}(m, F)$. Consider deleting a row r. Let $C_r(A)$ be the matrix that consists of the repeated columns of the matrix that is obtained when deleting row r from A. If we permute the rows of A so that r becomes the first row, then after some column permutations, A looks like this:
$$A = \begin{matrix} r \\ \\ \end{matrix}\begin{bmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ B_r(A) & & C_r(A) & C_r(A) & & D_r(A) \end{bmatrix}. \tag{2}$$

where $B_r(A)$ are the columns that appear with a 0 on row r, but don't appear with a 1, and $D_r(A)$ are the columns that appear with a 1 but not a 0. Prove that

$$\mathrm{forb}(m, F) \le |C_r(A)| + \mathrm{forb}(m-1, F). \tag{3}$$

4. Let K_k denote the $k \times 2^k$ simple $0-1$-matrix (configuration). Use the decomposition (2) and the inequality (3) to prove that $\mathrm{forb}(m, K_k) = O(m^{k-1})$.

5. Prove that
$$\mathrm{forb}(m, K_k) \ge \binom{m}{k-1} + \binom{m}{k-2} + \ldots + \binom{m}{0}. \tag{4}$$

6. Do we have equality in (4)?

7. Prove that
$$I_p \times T_p \in \mathrm{Avoid}(m, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}). \tag{5}$$