We will investigate the maximum size of the subsets of the vertices of a hypercube satisfying the property that the subspace (or cone) spanned by them will not intersect (contain) a given — other — subset of the vertices of the cube. It will turn out that the two cases when, on one side, we consider the spanned subspaces over $GF(2)$, i.e. work only inside the hypercube, and, on the other side, consider subspaces over $\mathbb{R}$, will yield different results. In the first case the maximal subsets with the required property will naturally form a subspace (unless we assume some further properties about the chosen vertices) by themselves — and we may not speak about a cone over $GF(2)$) —, while in the second case they will not necessarily do. For the more detailed, real case an interesting connection to the maximal size of subsums of a (positive) sum which are equal to 0 (in which case the numbers are supposed to be non-zero) or which are also positive, will be pointed out as well as a conjecture given related to the Littlewood-Offord and Erdős-Moser problems.

1. Introduction

Consider the different special cases of the following general question:

Question 1.1. How many vertices (maybe of a certain further property, e.g of fixed weight) of the n-dimensional hypercube can be picked such that their span, that is the subspace spanned by them — either over $GF(2)$ or over $\mathbb{R}$ — (or, in case of $\mathbb{R}$ the convex or positive span, that is, the cone spanned by them), does not contain or does not intersect certain configurations of the hypercube (vertieces, vertices of certain weight(s), subspaces or hyperplanes)?

Throughout the paper, C_n will denote the set of the vertices of the n-dimensional hypercube, $M \subset C_n$ will be a subset of it, $\operatorname{span}_2(M)$ will denote the subspace (of C_n) spanned by M over $GF(2)$, $\operatorname{span}_{\mathbb{R}}(M)$ will denote the subspace of $\mathbb{R}^n$ spanned by M (which is naturally not a subset of the hypercube, but contains some vertices, at least the vertices belonging to M) and $\operatorname{cone}(M) = \operatorname{cone}_{\mathbb{R}}(M)$ will denote the cone spanned by M (over $\mathbb{R}$), that is the collection of points of $\mathbb{R}^n$ obtained as a linear combination of the points from M with all positive coefficients. The elements (vertices) of the hypercube C_n will be considered as vectors (many times identified with the subset of $[n]$ having them as their characteristic vectors) and will be denoted by $\underline{x}_i$ with the vectors of exactly one, the i^{th} coordinate equal to 1 denoted by $\underline{e}_i = \{0, \dots, 0, 1, 0, \dots, 0\}$, and the whole 1 vector (vertex) denoted by $\underline{1} = \{1, 1, \dots, 1\}$.

The most obvious forms of the above general question are these: How big M can be such that $\underline{1} \notin \operatorname{span}(M)$ or $\underline{e}_1 = (1, 0, \dots, 0) \notin \operatorname{span}(M)$ or none of the vectors (vertices of the hypercube) $\underline{e}_i$ are in $\operatorname{span}(M)$, where span can mean either span_2 or $\operatorname{span}_{\mathbb{R}}$, or $\underline{1} \notin \operatorname{cone}(M)$? Moreover, naturally, the k-weighted versions of these questions can be asked as well, that is when we restrict the choices of the vertices of C_n only to those having weight k, i.e. exactly k coordinates equal to 1, all others being equal to 0.

It will turn out that the two most interesting questions of the above ones are equivalent to the following ones, respectively

Question 1.2. How should we give a set of n real numbers $x_1, x_2, \dots, x_n$ none of them being equal to 0, such that they maximize the number of sums $\sum_{i \in B} x_i$ equal to 0, where the B's are subsets of $\{1, 2, \dots, n\}$ (or B's are subsets of $\{1, 2, \dots, n\}$ of cardinality k or at most k)?

Question 1.3. Let $x_1, x_2, \dots, x_n$ be given numbers such that $\sum_{i=1}^n x_i > 0$. What is maximum number of negative subsums (or the minimum number of positive subsums) of exactly k of these numbers?

In section 2 of the paper we will explore the non-restricted cases (when the chosen vertices are not restricted to weight k vertices only) of the above questions and point out the connection of it to the Littlewood-Offord problem (basically to Question 1.2 above). It will also be shown that Question 1.2 is equivalent to the following question:

Question 1.4. Let $x_1, x_2, \dots, x_n$ be given numbers. What is the maximum number of the subsums $\sum x_{j_i}$ of them which can be disclosed without the disclosure of the values x_i, that is such a way that knowing those subsums nobody could calculate the values of the x_i's?

We will also discuss in this chapter the relation of $\mathrm{span}_2(M)$ and $\mathrm{span}_{\mathbb{R}}(M)$ for a given subset M of vertices of the hypercube.

In section 3 the k-weight cases will be inspected, in particular the relation of them to Question 1.3 and to the following question as well:

Question 1.5. Let $x_1, x_2, \ldots, x_n$ be given numbers. What is the maximum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ (of exactly k of or at most k of these numbers) which can be given without the disclosure of the values of x_i's?

It will be pointed out that the answers to the questions have many times kind of "phase transition" phenomena, that is in lower dimension they will be different from the ones in higher dimension.

Finally, in section 4 we will return to the non-restricted case and investigate, or at least pose some further questions, where the subsets of the vertices of the hypercube to be avoided will be of other nature, like all vertices of weight 2, or similar. An interesting conjecture will be formulated for this case, related to the Erdős-Moser problem.

It was brought to the attention of the author during the preparation of this survey paper that during recent years R. Ahlswede, H. Aydinian, and L. H. Khachatrian wrote a series of papers [1, 2, 3, 4, 5] dealing with similar questions and especially [2, 3] contain results stated in this paper as well, though mostly from a different approach and most of the time with different proofs.

2. The general questions

Proposition 2.1. *If for an $M \subset C_n$ the size of $M > 2^{n-1}$ then $(1, 0, \ldots, 0) \in span(M)$, and therefore $(1, 1, \ldots, 1) \in span(M)$.*

Proof. It is easy to see that $|M| > 2^{n-1}$ implies that two "complementary" pair of vertices, that is, two vertices whose sum as vectors equals $(1, 1, \ldots, 1)$, should be in M, and therefore $(1, 1, \ldots, 1) \in \mathrm{span}(M)$. The fact that $|M| > 2^{n-1}$ implies $(1, 0, \ldots, 0) \in \mathrm{span}(M)$ will be shown by induction on n. The base cases ($n = 2$ or 3) are easy to check.

Divide M into two disjoint subsets, M_1 being the set of vertices with last coordinates 0, and M_2 the set of vertices with last coordinates 1. Since $|M| > 2^{n-1}$, either $|M_1| > 2^{n-2}$ or $|M_2| > 2^{n-2}$. In the first case, by the induction hypothesis, the vector $(1, 0, \ldots, 0)$ of length $n-1$ is in the span of the vectors obtained by truncating the last 0 coordinate from the vertices of M_1, and therefore the vector $\underline{e}_1 = (1, 0, \ldots, 0)$ of length n is in the span of M_1. In the second case, knowing already that $\underline{1} = (1, 1, \ldots, 1) \in \mathrm{span}(M)$, subtract the vectors of M_2 from this vector, obtaining more than 2^{n-2} vectors in span M with last coordinate equal to 0, thus reducing this case to the first one.

Note that the above arguments work for both of span_2 and $\mathrm{span}_{\mathbb{R}}$, therefore this is proposition is valid, independently whether we work over $GF(2)$ or $\mathbb{R}$.

Also, the same arguments show the validity of the following two propositions as well (in case of Proposition 2.2, again, independently whether we take the span over $GF(2)$ or $\mathbb{R}$).

Proposition 2.2. *If for an $M \subset C_n$ the size of $|M| > 2^{n-1}$ then $span(M) \supset C_n$.*

Proposition 2.3. *If for an $M \subset C_n$ the size of $|M| > 2^{n-1}$ then $(1, 1, \ldots, 1) \in cone(M)$.*

Remark 2.4. All the above bounds are sharp, since for any $x \in C_n$, $x \neq \underline{0}$ one can take a non-zero coordinate of x, and M as the set of all (2^{n-1}) vertices of C_n which have 0 at this coordinate. The span of this M will obviously not contain x (neither in $GF(2)$ nor in $\mathbb{R}$). This construction is valid for the vertex $(1, 1, \ldots, 1)$ and for positive span (cone) as well.

Remark 2.5. The question about the maximum size of M with $\operatorname{span} M$ not containing completely any given subset B of C_n is handled by the above propositions: Any vertex of C_n, in particular any vertex of B can be "avoided" by an M of size 2^{n-1} (that is, $\operatorname{span}M$ will not contain that vertex, therefore will not contain B completely). On the other hand, any M of size greater than 2^{n-1} will have $\operatorname{span}M$ containing the whole C_n, in particular, $\operatorname{span}M$ will contain B completely. Again, in the argument above span may be meant both as span_2 or $\operatorname{span}_{\mathbb{R}}$.

The situation becomes more diverse in case we want to avoid with the span of M *all* vertices $\underline{e}_i$.

Theorem 2.6. *The maximum size of a subset M of the vertices of C_n such that none of the vertices $\underline{e}_i = (0,0,\dots,0,1,0,\dots,0)$ are in $\operatorname{span}_{\mathbb{R}}(M)$ is $\binom{n}{\lfloor n/2\rfloor} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{i} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{\lfloor n/2\rfloor-i}$*

(as shown by taking all vertices having the same number of 1 coordinates among the first $\lfloor n/2\rfloor$ and last $\lceil n/2\rceil$ coordinates).

In case of span_2 the above bound may not be valid (since in that case we have M a subspace of C^n, definitely having size of a power of 2), and so the situation is quite different, as the following remark shows.

Remark 2.7. The maximum size of a subset M of the vertices of C_n such that none of the vertices $(0,0,\dots,0,1,0,\dots,0)$ are in $\operatorname{span}_2(M)$ is still 2^{n-1}, as shown by the following example: divide the set of coordinates into two arbitrary subsets A and B (in extreme case $|B|=1$) and take all of those vertices of C_n which have the same parity of 1 coordinates in A and B (in case of $|A| = 2^{n-1}$ and $|B|=1$ it simply means taking all the vertices with arbitrary coordinates in A and the only coordinate belonging to B being the parity check bit of them). All linear combinations over $GF(2)$ will preserve the parity of the 1 coordinates both in A and B, therefore no vertices with exactly one coordinate equal to 1 will be among them. The previous propositions show that this is the best possible construction.

Remark 2.8. Though it is clear that for any subset M of the vertices of C_n $\dim \operatorname{span}_2 M \le \dim \operatorname{span}_{\mathbb{R}} M$, we can not state any containment relation between $\operatorname{span}_2 M$ and $\operatorname{span}_{\mathbb{R}} M \cap C_n$. If one chooses M_1 to be the set of vertices (of dimension 4) $\{(0000),(1111),(1010),(1001),(0110),(0101),(1100),(0011)\}$, then — as it can be easily seen — $\operatorname{span}_2 M_1 = M_1$ over $GF(2)$, while $\operatorname{span}_{\mathbb{R}} M_1 = \mathbb{R}^4$, showing therefore that $\operatorname{span}_2 M \not\supset \operatorname{span}_{\mathbb{R}} M \cap C_n$ (in general). On the other hand, choosing $M_2 = \{(0000),(1111),(1010),(1001),(0110),(0101)\}$, we get that $\operatorname{span}_2 M_2 = M_1$, while $\operatorname{span}_{\mathbb{R}} M_2$ will not contain any further vertices of C_n, therefore showing that $\operatorname{span}_2 M \not\subset \operatorname{span}_{\mathbb{R}} M \cap C_n$ (in general).

Clearly, Theorem 2.6 is equivalent to the following theorem:

Theorem 2.9. *(Miller at al. 1991* [14, 15], *Griggs, 1997* [13]*)*
Let $x_1, x_2, \dots, x_n$ be given numbers. The maximum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ which can be given without the disclosure of the values x_i is $\sum_{i=0}^{\lfloor n/2\rfloor}\binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{i} = \sum_{i=0}^{\lfloor n/2\rfloor}\binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{\lfloor n/2\rfloor-i} = \binom{n}{\lfloor n/2\rfloor}$

(as shown by taking all subsums having the same number of elements among the first $\lfloor n/2\rfloor$ and last $\lceil n/2\rceil$ elements).

Indeed, if we take the characteristic vectors of the subsums in Theorem 2.9 (and view them as vertices of C_n), their $\operatorname{span}_{\mathbb{R}}$ must avoid all vertices of C_n with exactly one 1 coordinate (otherwise the corresponding $\underline{e}_i$ would be a linear combination of the disclosed sum, therefore easily computable). On the other hand, if we have a subset M of the vertices of C_n such that $\operatorname{span}_{\mathbb{R}} M$ does not contain any $\underline{e}_i \in C_n$, consider a solution to the equation $\underline{x}\mathbf{M} = \underline{b}$ where $\mathbf{M}$ is the matrix consisting of columns formed by the element of M as vectors, $\underline{b}$ is the vector consisting of the supposedly disclosed values of the subsums corresponding to the vectors in M and therefore $\underline{x}$ consisting of coordinates which might be the possible values of the x_i's, giving exactly these values of the subsums. Now, $\underline{e}_i = \{0,0,\dots,1,\dots,0\}$, having exactly and only the i^{th} coordinate equal to 1, is not in $\operatorname{span}_{\mathbb{R}} M$, therefore $\underline{e}'_i$, the component of it perpendicular to the subspace $\operatorname{span}_{\mathbb{R}} M$ has non-zero value in the i^{th} coordinate. That is, $(\underline{x} + \underline{e}'_i)$ is another solution of the equation $(\underline{x} + \underline{e}'_i)\mathbf{M} = \underline{b}$ and has a different value in the i^{th} coordinate, showing that it is impossible to calculate the value of x_i from the disclosed values of the subsums.

In [7] or [8, 9] it was proven that the above two theorems are equivalent to the following one as well.

Theorem 2.10. *Given a set of n real numbers $x_1, x_2, \ldots, x_n$ none of them being equal to 0, the maximum number of sums $\sum_{i\in B} x_i$ equal to 0, where the B's are subsets of $\{1, 2, \ldots, n\}$ is*

$$\sum_{i=0}^{\lfloor n/2\rfloor} \binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{i} = \sum_{i=0}^{\lfloor n/2\rfloor} \binom{\lceil n/2\rceil}{i}\binom{\lfloor n/2\rfloor}{\lfloor n/2\rfloor - i} = \binom{n}{\lfloor n/2\rfloor}$$

(as shown by taking $\lfloor n/2\rfloor$ 1's and $\lceil n/2\rceil$ -1's).

This theorem has a similarity with the famous Littlewood-Offord problem:

Problem 2.11. (Littlewood-Offord) How do we select — not necessarily distinct — complex numbers $x_1, x_2, \ldots x_n$, with $|x_i| \geq 1$ and an open unit diameter ball B, such that they maximize the number out of the 2^n sums $\sum_{i\in I} a_i$, $I \subseteq [n]$, lying inside B?

The following simpler version of it, considering only reals, instead of complex numbers, i.e., vectors of dimension two, was first solved by Erdős [10]. His argument can be used to prove Theorem 2.10, more precisely, the following, a bit more general statement:

Theorem 2.12. *Given a set of n real numbers $x_1, x_2, \ldots, x_n$ none of them being equal to 0, the maximum number of sums $\sum_{i\in B} x_i$ equal to a fixed t (where the B's are subsets of $\{1, 2, \ldots, n\}$) is $\binom{n}{\lfloor n/2\rfloor}$.*

Proof (though it can be found at several papers, we include here the following, simplest argument, based on Griggs, copying proof of Erdős' for the Littlewood-Offord problem and even further simplified by Cameron [6]). Divide the set of indices $\{1, 2, \ldots, n\}$ into two parts, the indices of the negative and the positive x_i's: $M = N \cup P$. Assign to any subset of the indices B a new subset $B' = (N \cap B) \cup (\overline{N} \cap P)$. Clearly, $B'_1 \subset B'_2$ yields that $\sum_{i\in B_1} x_i < \sum_{i\in B_2} x_i$, therefore the subset of the indices yielding the same subsum value form a Sperner system, their number may not exceed $\binom{n}{\lfloor n/2\rfloor}$.

Remark 2.13. The questions about the maximum size of M with cone M not containing any or all of the vertices of the form $\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ are trivial, therefore not interesting. (A cone spanned by the vertices of the cube will contain a vertex of type $\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ iff the vertex $\underline{e}_i$ is among the spanning vertices.)

3. The weight k restricted case

Again, as throughout in the paper, let C_n denote the vertices of the n dimensional hypercube and let $M_k \subset C_n$ be a subset of it consisting of vertices of weight k only (vertices with exactly k coordinates equal to 1). In this chapter we will discuss the following general question:

Question 3.1. How big M_k can be such that $(1, 0, 0, \ldots, 0) \notin \text{span}(M_k)$ or $(1, 1, \ldots, 1) \notin \text{span}(M_k)$ or $(1, 1, \ldots, 1) \notin \text{cone}(M_k)$ (again, span denoting the general question or answer for both of the cases over $GF(2)$ or $\mathbb{R}$, while span_2 or $\text{span}_{\mathbb{R}}$ will stand for the specific cases, but the cone case is considered only over $\mathbb{R}$).

To obtain lower bounds, consider the following sets M_k of vertices of C_n of weight k

1. vertices with first coordinate $= 0$ (the number of them is $\binom{n-1}{k}$ and their span (cone, in case of $\mathbb{R}$) will definitely contain neither $(1, 0, 0, \ldots, 0)$ nor $(1, 1, \ldots, 1)$).
2. vertices with last coordinate $= 1$ (the number of them is $\binom{n-1}{k-1}$ and their span over $\mathbb{R}$ or even over $GF(2)$ will not contain $(1, 0, 0, \ldots, 0)$, while their span over $\mathbb{R}$ will not contain $(1, 1, \ldots, 1)$ either).

3. vertices with exactly one of the last two coordinates $= 1$, and having the remaining $k-1$ 1 coordinates chosen from the remaining $n-2$ positions (the number of them is $2\binom{n-2}{k-1}$ and their span over $\mathbb{R}$ or $GF(2)$ will not contain $(1,0,0,\ldots,0)$, and their span over $\mathbb{R}$ will not contain $(1,1,\ldots,1)$ either, provided $n \neq 2k$).

In case 2, assume that we are given values $x_1, \ldots x_n$ and some subsums of k of them — having characteristic vectors equal to the given vertices in M_k — are disclosed, all containing x_n. Increase the value of x_n by $k-1$ and decrease all others by 1, therefore the given subsums will remain, though each of the values and the sum of them are changed, that is, neither x_1 nor $\sum_{i=0}^{n} x_i$ can be calculated from the given subsums, and therefore none of $(1,0,0,\ldots,0)$ and $(1,1,\ldots,1)$ can be in $\mathrm{span}_{\mathbb{R}} M_k$. A similar argument works for case 3, increasing the values of the variables corresponding to the first two coordinates each by $k-1$ and decreasing all other values by 1. However, only in case of $n \neq 2k$ will this last argument work for the vector $(1,1,\ldots,1)$ since decreasing the first $2k-2$ elements by 1 and increasing the last two by $k-1$ will not change the total sum of the numbers in case of $n = 2k$.
In case of $GF(2)$ and case 2, assuming that $(1,0,0,\ldots,0) = \underline{e}_1 \in \mathrm{span}_2 M_k$ the number of vectors in the linear combination giving $\underline{e}_1$ must be even (since the last coordinate must be 0 in the linear combination), resulting an even number of 1's in total, not allowing $\underline{e}_1$ in $\mathrm{span}_2 M_k$. A similar argument works for case 3, only now the number of vectors in the linear combination yielding $\underline{e}_1$ having last coordinate equal to 1 must be even, as well as the number of vectors having the last but one coordinate equal to 1. Since we only consider vectors with exactly one of the last two coordinates equal to 1, these two collections of vectors are disjoint, giving in total an even number vectors, and therefore an even number of "1"'s in the linear combination, which, therefore, may not give $\underline{e}_1$. Therefore, for every n and k one can choose a set of at least $\max\left\{\binom{n-1}{k}, \binom{n-1}{k-1}, 2\binom{n-2}{k-1}\right\}$ vectors of weight k (to form our M_k) such that neither $\mathrm{span}_2 M_k$ nor $\mathrm{span}_{\mathbb{R}} M_k$ contain $\underline{e}_1$. Also, we can choose a set of at least $\binom{n-1}{k}$ vectors of weight k (to form our M_k) such that $\mathrm{span}_2 M_k$ does not contain $\underline{1} = \{1,1,\ldots,1\}$ by construction number 1 above. However, construction number 2, giving a lower bound of $\binom{n-1}{k-1}$ as well, may or may not work, depending on the actual values of n and k. For example, if the parity of n and k are different, a parity argument will show that the span_2 of the set M_k obtained by construction 2 will still not contain $\underline{1}$, while if both $\binom{n-1}{k-1}$ and $\binom{n-2}{k-2}$ are odd, simply the sum of all vectors with last coordinate equal to 1 will give $\underline{1}$.

Remark 3.2.

$$\max\left\{\binom{n-1}{k}, \binom{n-1}{k-1}, 2\binom{n-2}{k-1}\right\} = \begin{cases} \binom{n-1}{k-1} & \text{for } n \leq 2k-2 \\ \binom{n-1}{k-1} = 2\binom{n-2}{k-1} & \text{for } n = 2k-1 \\ 2\binom{n-2}{k-1} & \text{for } n = 2k \\ \binom{n-1}{k} = 2\binom{n-2}{k-1} & \text{for } n = 2k+1 \\ \binom{n-1}{k} & \text{for } n \geq 2k+2 \end{cases}$$

and

$$\max\left\{\binom{n-1}{k}, \binom{n-1}{k-1}\right\} = \begin{cases} \binom{n-1}{k-1} & \text{for } n \leq 2k-1 \\ \binom{n-1}{k-1} = \binom{n-1}{k} & \text{for } n = 2k \\ \binom{n-1}{k} & \text{for } n \geq 2k+1 \end{cases}$$

Let $m_1 = m_1^{(n,k)} = \max\left\{\binom{n-1}{k}, \binom{n-1}{k-1}, 2\binom{n-2}{k-1}\right\}$ and $m_2 = m_2^{(n,k)} = \max\left\{\binom{n-1}{k}, \binom{n-1}{k-1}\right\}$. These two numbers will be the exact bounds for the $\mathbb{R}$ case. Unfortunately the case of $GF(2)$ is much more complicated, where further parity constraints must be considered, since, e.g., in case of k being even there is no way to get a linear combination of vectors of weight k giving $\underline{e}_1$, a vector of weight 1, that is odd weight.

Theorem 3.3. *If for an $n \geq k$ and $M_k \subset C_n$ the size of $M_k > m_1$ then $(1,0,\ldots,0) \in \mathrm{span}_{\mathbb{R}}(M_k)$, and therefore $\mathrm{span}_{\mathbb{R}}(M_k) = C_n$.*

Proof goes by induction on n, and then for a fixed n, by induction on k, that is, to prove the validity of the statement for (n,k), we will assume it is true for every (n',k') with either $n' < n$ or in case of $n' = n$ with $k' < k$. The base cases are $k = 1$ (trivial) and then for every bigger k we will also need the $n = k+1$ case, when trivially $\binom{n-1}{k-1} = \binom{k}{k-1} = k$ is the right bound (having more than that many vertices would include *all* vertices of weight k and therefore any vertex of weight 1 — and, as a consequence, all other vertices — would be in $\mathrm{span}_{\mathbb{R}}$).

For the inductional step (in general), assume that $|M_k| > m_1 = m_1^{(n,k)}$ and will prove that $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$. Let M_k be partitioned into two subsets, M_k^i, the sets of vertices from M_k having their last coordinates equal to i, $i = 0, 1$. In case $|M_k^0| > m_1^{(n-1,k)}$ we will trivially have $(1, 0, \ldots, 0) \in \mathrm{span}(M_k)$. Similarly, in case of $|M_k^1| > m_1^{(n-1,k-1)}$ a linear combination of the vertices obtained from the vertices of M_k^1 considering only the first $n-1$ coordinates (and having $k-1$ 1 coordinates there) will be of the form $(1, 0, \ldots, 0)$. The same linear combination on the last, n^{th} coordinates (all of them being 1) will result a_1 and therefore we will have a vertex of the form $\underline{v}_1 = (1, 0, \ldots, 0, a_1) \in \mathrm{span}(M_k)$. Similarly, for every $1 \le i \le n-1$ we will have a vertex of form $\underline{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0, a_i) \in \mathrm{span}(M_k)$, where the only 1 coordinate is at position i. An easy counting of the total weights of the linear combination on the first $n-1$ and, independently, on the last, n^{th} coordinate will give that $a_i = \frac{1}{k-1}$ for every i. Summing up the appropriate choice of $k-1$ of these vertices will give us all of those vertices of weight k which have the last coordinate equal to 1, in $\mathrm{span}_{\mathbb{R}} M_k$. Assume there is another vertex of M_k, that is, another one from M_k^0, with last coordinate equal to 0, say $\underline{v}$. Change the first 1 coordinate — assume it's position is at i — of $\underline{v}$ to 0, and the very last coordinate (which was supposed to be 0) to 1, resulting a vertex already in $\mathrm{span}_{\mathbb{R}} M_k$. Take the difference of them, $(0, \ldots, 0, 1, 0 \ldots, 0, -1)$ and consider the earlier $v_i = (0, \ldots, 0, 1, 0, \ldots, 0, a_i)$, an appropriate linear combination of which giving $(0, \ldots, 0, 1)$, or, equivalently, $(0, \ldots, 0, \frac{1}{k-1})$. Subtracting this from $\underline{v}_1$ will result $\underline{e}_1 = (1, 0, \ldots, 0)$, the sought vertex in $\mathrm{span}_{\mathbb{R}} M_k$.

Therefore, we only need to check that for every $n \ge k+2$ and set of vertices M_k of dimension n and weight k with $|M_k| > m_1 = m_1^{(n,k)}$ either *(a)* $|M_k^0| > m_1^{(n-1,k)}$ or *(b)* $|M_k^1| > m_1^{(n-1,k-1)}$ *and we have a vertex in* M_k^0 *as well.* For that, it will be enough to prove that $m_1^{(n,k)} \ge m_1^{(n-1,k)} + m_1^{(n-1,k-1)}$, since the second condition is automatically ensured by the fact that $|M_k^1| \le \binom{n-1}{k-1} \le m_1^{(n,k)} < |M_k|$, and therefore a vertex from M_k must be outside of M_k^1, that is, in M_k^0.

The inequality

$$m_1^{(n,k)} \ge m_1^{(n-1,k)} + m_1^{(n-1,k-1)} \tag{1}$$

is almost always true and should be checked for all possible values of (n, k), together with the missing inductional steps when (1) does not hold. Again, assume that $|M_k| > m_1 = m_1^{(n,k)}$

(i) $n \le 2k-2$ in which case $m_1^{(n,k)} = \binom{n-1}{k-1}$, $m_1^{(n-1,k)} = \binom{n-2}{k-1}$ and $m_1^{(n-1,k-1)} = \binom{n-2}{k-2}$ and therefore (1) is true, $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$.

(ii) $n = 2k-1$ in which case consider $\overline{M_k} = \{\underline{1} - a \colon a \in M_k\}$, a set of vertices of the hypercube C_{2k-1} of weight $k-1$. By the induction hypothesis we know that if $(|M_k| =) |\overline{M_k}| > m_1^{(2k-1,k-1)} = \binom{2k-2}{k-1} = m_1^{(2k-1,k)}$ then both $\underline{e}_1$ and $\underline{1}$ are in $\mathrm{span}_{\mathbb{R}} \overline{M_k}$, that is, there are linear combinations of the vectors from M_k such that $\sum d_i(\underline{1} - \underline{a}_i) = \underline{1}$ and $\sum c_i(\underline{1} - \underline{a}_i) = \underline{e}_1$. From the first equation we have $\sum d_i \underline{a}_i = (\sum d_i - 1)\, \underline{1}$, where again an easy calculation of the weights shows that $\sum d_i = \frac{n}{n-k}$ and therefore $\underline{1} \in \mathrm{span}_{\mathbb{R}} M_k$. From the second equation we get that $\sum c_i \underline{1} - \sum c_i \underline{a}_i = \underline{e}_1$, that is $\underline{e}_1 = \sum -c_i \underline{a}_i + (\sum c_i)\, \underline{1}$, yielding $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$.

(iii) $n = 2k$ in which case $m_1^{(n,k)} = 2\binom{n-2}{k-1}$, $m_1^{(n-1,k)} = \binom{n-2}{k-1}$ and $m_1^{(n-1,k-1)} = \binom{n-2}{k-1}$ and therefore (1) is true, $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$.

(iv) $n = 2k+1$, in which case we assume that $|M_k| > m_1 = m_1^{(2k+1,k)} = \binom{2k}{k}$. First assume that $|M_k^1| > m_1(2k, k-1) = \binom{2k-1}{k-1}$, in which case the general induction step described at the beginning of the prof works. Otherwise we may assume that $|M_k^0| \ge \binom{2k-1}{k} = \frac{1}{2}\binom{2k}{k}$, which is unfortunately not enough to prove that $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$, but gives us two "complementary" vectors in the first $2k$ coordinates, giving $\underline{b} = (1, 1, 1, \ldots, 1, 1, 0) \in \mathrm{span}_{\mathbb{R}} M_k$. Consider $M'^1_k = \{\underline{b} - \underline{a} \colon \underline{a} \in M_k^1\} \subset \mathrm{span}_{\mathbb{R}} M_k$, a set of vectors of dimension $2k+1$ with exactly $k+1$ "1" coordinates among the first $2k$ coordinates,

-1 as the last coordinate and all other coordinates equal to 0. By induction we know that in case the number of these vectors (equal to $|M'^1_k| = |M^1_k|$) is more than $m_1(2k, k+1) = \binom{2k-1}{k+1}$, an argument similar to the general inductional step will show that for every $1 \le i \le n-1$ there is a vertex of form $\underline{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0, a_i) \in \mathrm{span}_{\mathbb{R}}(M_k)$, where the only 1 coordinate is at position i, and a similar counting of the total weights of the linear combination on the first $n-1$ and, independently, on the last, n^{th} coordinate will give that $a_i = \frac{-1}{k+1}$ for every i. By an earlier comment still there must be a vector, say $\underline{b} \in M^0_k$. Take all of those $\underline{v}_i$'s which have "1" coordinates at the positions of the "1" coordinates of $\underline{b}$, sum them up, resulting a vector only different from $\underline{b}$ in the last coordinate (where $\underline{b}$ has 0,while the sum obviously $-\frac{k}{k+1}$). It gives that $\{0, 0, \ldots, -\frac{k}{k+1}\} \in \mathrm{span}_{\mathbb{R}} M_k$, which together with $\underline{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0, a_i) \in \mathrm{span}_{\mathbb{R}}(M_k)$ gives that $\underline{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0, 0) \in \mathrm{span}_{\mathbb{R}}(M_k)$ for every i.

(v) $n \ge 2k+2$ in which case $m_1^{(n,k)} = \binom{n-1}{k}$, $m_1^{(n-1,k)} = \binom{n-2}{k}$ and $m_1^{(n-1,k-1)} = \binom{n-2}{k-1}$ and therefore (1) is true, $\underline{e}_1 \in \mathrm{span}_{\mathbb{R}} M_k$. ∎

Theorem 3.4. *If for an $M_k \subset C_n$ the size of $M_k > m_2$ then $(1, 1, \ldots, 1) \in span(M_k)$.*

Proof Note that $m_1(n, k) = m_2(n, k)$ almost always, with the only exception of $n = 2k$. However, in this last case $m_2(2k, k) = \binom{2k-1}{k} = \binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}$ and assuming $|M_k| > m_2(n, k)$ in this case gives us two "complementary" vectors in M_k, sum of which is exactly equal to $(1, 1, \ldots, 1)$. This, together with the constructions at the beginning of this chapter plus Theorem 3.2 completes the proof. ∎

It is natural to ask whether the result of Theorem 3.3 remains valid if we restrict ourselves to convex combinations of the vectors in M_k, that is, which value of $|M_k|$ will surely result that $(1, 1, \ldots, 1, 1) \in \mathrm{cone} M_k$. Note here that this question is only meaningful over $\mathbb{R}$ and even in that case meaningless for $(1, 0, \ldots, 0, 0)$, since a cone contains a vertex of type $(1, 0, \ldots, 0, 0)$ iff the vertex is among the vertices spanning the cone.

The situation becomes more complicated, as the following example shows that for certain $n > 2k$ we have a set of k-uniform subsets (set of vertices of weight k) $M_k \subset C_n$ of size bigger than $m_2(n, k) = \binom{n-1}{k}$ with $(1, 1, \ldots, 1, 1) \notin \mathrm{cone} M_k$.

Let $n = 3k+1$ and consider M_k, the set all of those vertices of C_n of weight k which have at least one of the first three coordinates equal to 1. In this case we have $\binom{3k-2}{k}$ vertices of weight k having the first three coordinates 0, which is less than $\binom{3k}{k-1}$, and therefore the number of vertices having at least one of the first three entries equal to 1, $\binom{3k+1}{k} - \binom{3k-2}{k}$, is more than $\binom{3k+1}{k} - \binom{3k}{k-1} = \binom{3k}{k} = \binom{n-1}{k}$.

We claim that the cone spanned by M_k will not contain $(1, 1, \ldots, 1)$. The simplest way to see it is to give an alternative form of the question above and consider the example in that framework:

Question 3.5. Let $x_1, x_2, \ldots, x_n$ be given numbers such that $\sum_{i=1}^n x_i > 0$. What is maximum number of negative subsums (or the minimum number of positive subsums) of exactly k of these numbers?

It is obvious that for certain choices of $x_1, x_2, \ldots, x_n$ one will have at least $\binom{n-1}{k}$ negative k-subsums, shown by the example of many small (absolute value) negative numbers and one big (absolute value) positive number, e.g. $\{-1, \ldots, -1, n\}$. Therefore, the minimum number of positive subsums is at most $\binom{n-1}{k-1}$ and in Question 3.5 the bound is at least as big as $\binom{n-1}{k}$, but maybe bigger sometimes, shown by the following example, equivalent to the above one about the cone not containing $(1, 1, \ldots, 1, 1)$:

Consider $3k+1$ numbers: $\{2-3k, 2-3k, 2-3k, 3, 3, \ldots, 3\}$ whose sum is 1. In this case there are $\binom{3k-2}{k}$ positive subsums, which is less than $\binom{3k}{k-1}$, and therefore $\binom{3k+1}{k} - \binom{3k-2}{k}$ negative subsums, which is more than $\binom{3k+1}{k} - \binom{3k}{k-1} = \binom{3k}{k} = \binom{n-1}{k}$.

To see the analogy between the two questions one may use a linear algebraic argument similar to the earlier cases: Assume the cone spanned by the vertices of C_n of weight k which have at least one of the first three coordinates equal to 1. To each of these vertices assign the k-subsum of the set $\{2-3k, 2-3k, 2-3k, 3, 3, \ldots, 3\}$ with having the vertex as it's characteristic vector. Since all of these sumbsums are negative, it may not happen that a linear combination of them will give the vector $(1, 1, \ldots, 1, 1)$ corresponding to the total, therefore positive sum.

Although the previous example shows that the bound might be bigger than $\binom{n-1}{k}$ in some cases, the following theorem still shows that the general bound will be this number.

Theorem 3.6. *(Manickam, Miklós, 1989) Let $x_1, x_2, \ldots, x_n$ be given numbers such that $\sum_{i=1}^{n} x_i > 0$. The minimum number of positive subsums of exactly k of these numbers is $\binom{n-1}{k-1}$ if $n > n_1(k)$ or k divides n.*

Corollary 3.7. *If for an $M_k \subset C_n$ the size of $M_k > \binom{n-1}{k}$ and either $n > n_1(k)$ or k divides n, then $(1, 1, \ldots, 1) \in cone(M_k)$.*

Next, we may ask the question analogous to the one asked for the non-restricted case (and we will restrain only to the real case here as well), that is

Question 3.8. What is the maximum size of a subset M_k of the vertices of the n-dimensional hypercube (all of weight k or weight at most k) such that none of the vertices of the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$ are in $\mathrm{span}_{\mathbb{R}}(M_k)$?

Similar to the non-restricted case, this question is proven to be equivalent in [8, 9] to the following two questions:

Question 3.9. Let $x_1, x_2, \ldots, x_n$ be given numbers. What is the maximum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ (of exactly k of or at most k of these numbers) which can be given without the disclosure of the values x_i?

and

Question 3.10. Let $x_1, x_2, \ldots, x_n$ be given numbers. What is the maximum number of the subsums $\sum_{i=1}^{k} x_{j_i}$ (of exactly k of or at most k of these numbers) being equal to 0?

Let the answer to these 3 questions be defined by $M(n, k)$ (in case of M_k consisting of vectors all of weight k), and $N(n, k)$ (in case of vectors of weight at most k), resp.

One may see that any number of the form $m_1(n_1, n_2, k_1, k_2) = \binom{n_1}{k_1}\binom{n_2}{k_2}$ with $n_1 + n_2 = n$ and $k_1 + k_2 = k$, and, independently,

$$m_2(n, k) = \sum_{i=1}^{\frac{k}{2}} \binom{\lceil \frac{n}{2} \rceil}{i} \binom{\lfloor \frac{n}{2} \rfloor}{i}$$

are lower bounds for $N(n, k)$, and $m_1(n_1, n_2, k_1, k_2)$ is a lower bound for $M(n, k)$ as well. For the bound $m_1(n_1, n_2, k_1, k_2)$ and Question 3.8 take M_k as the set of vertices of the hypercube containing exactly k_1 "1" coordinates among the first n_1 coordinates (and therefore exactly k_2 "1" coordinates among the last n_2 coordinates, for Question 3.9 subsums containing exactly k_1 x_i's among the first n_1 given numbers (and therefore exactly k_2 x_i's among the last n_2 given numbers), and for Question 3.10 consider n_1 copies of k_2 and n_2 copies of $-k_1$ (the checking of the validity of the fact these samples will have the required properties is based on some previous argument in this paper and left to the reader). For the bound $m_2(n, k)$ and Question 3.8 take M_k as the set of vertices of the hypercube containing exactly i "1" coordinates both among the first $\lceil \frac{n}{2} \rceil$ and last $\lfloor \frac{n}{2} \rfloor$ coordinates for $1 \le i \le \lfloor \frac{k}{2} \rfloor$, for Question 3.9 subsums containing exactly i x_i's both among the first $\lceil \frac{n}{2} \rceil$ and last $\lfloor \frac{n}{2} \rfloor$ given numbers (again, for $1 \le i \le \lfloor \frac{k}{2} \rfloor$), and for Question 3.10 consider $\lceil \frac{n}{2} \rceil$ copies of 1 and $\lfloor \frac{n}{2} \rfloor$ copies of -1. The checking is again easy and left to the reader.

In case of $m_1(n_1, n_2, k_1, k_2)$ these numbers give many different lower bounds. In order to get the best one, we need to maximize $\binom{n_1}{k_1}\binom{n_2}{k_2}$ for $n_1 + n_2 = n$ and $k_1 + k_2 = k$. A somewhat surprising result - since it shows that the maximum is reached at a rather marginal point - is in [8, 9]:

Theorem 3.11. *(Demetrovics, Katona, Miklós, 2004) Suppose $4 \leq k$, $n_1(k) \leq n$. The maximum size of a subset M_k of the vertices of the n-dimensional hypercube (all of weight at most k) such that none of the vertices of the form $(0,0,\ldots,0,1,0,\ldots,0)$ are in $\mathrm{span}_{\mathbb{R}}(M_k)$ is*

$$N(n,k) = \binom{\left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor}{k-1} \left(n - \left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor\right)$$

which answer is of the form $\binom{n_1}{k_1}\binom{n_2}{k_2}$ with $n_1 = \left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor$, $n_2 = n - \left\lfloor \frac{(n+1)(k-1)}{k} \right\rfloor$, $k_1 = k-1$, $k_2 = 1$, therefore also is the answer for $M(n,k)$, the case with vectors all of weight exactly equal to k. This later result, even without the assumption that $n_1(k) \leq n$ was also obtained by Ahlswede, Aydinian, and Khachatrian in [3].

On the other hand, the assumption of $n_1(k) \leq n$ is necessary for the case of $N(n,k)$, when the sizes of the chosen subsets (or, equivalently, weight of the chosen vertices) are only bounded above by k, not necessarily equal to k. For example, as Theorems 2.9 shows,

$$N(n,n) = \binom{n}{\lfloor n/2 \rfloor} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{\lceil n/2 \rceil}{i}\binom{\lfloor n/2 \rfloor}{i} = m_2(n,n)$$

and not $m_1(n_1,n_2,k_1,k_2)$ for ceratin values of the parameters.

It is expected that the same construction remains the best if n is not much larger than k, that is (assuming for convenience that k is even)

$$N(n,k) = \sum_{i=1}^{\frac{k}{2}} \binom{\lceil \frac{n}{2} \rceil}{i}\binom{\lfloor \frac{n}{2} \rfloor}{i}.$$

For example, it is known that $N(12,6) = \binom{6}{3}\binom{6}{3} + \binom{6}{2}\binom{6}{2} + \binom{6}{1}\binom{6}{1} = m_2(12,6)$, however, $N(20,6) = M(20,6) = \binom{17}{5} \cdot 3 = m_1(17,3,5,1) = \max m_1(n_1,n_2,k_1,k_2)$ with $n_1 + n_2 = 20$ and $k_1 + k_2 = 6$, like in Theorem 3.11.

4. Further questions — back to the unrestricted case

Returning to the unrestricted case, we may ask, for example, for the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight $n-1$, or, using the same translation we had earlier, given n real numbers $x_1, x_2, \ldots, x_n$, find the maximum number of subset sums $\sum_{i \in B} x_i$ equal to 0, where the B's are subsets of $\{1,2,\ldots,n\}$, with the condition that none of the sums $(\sum_{i=1}^n x_i) - x_j$ are equal to 0, that is no sum of $n-1$ of the given numbers is equal to zero. Or, in general, find the maximum number of subset sums $\sum_{i \in B} x_i$ equal to 0, for any set of n real numbers $x_1, x_2, \ldots, x_n$, with the condition that none of the sums $\sum_{j=1}^r x_{i_j}$ are equal to 0 (for a given r). (For $r = n$ the answer is 2^{n-1} by Proposition 2.1 and for $r = 1$ it is $\binom{n}{\lfloor n/2 \rfloor}$ by Theorem 2.6). This later will correspond to finding the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight r.

We prove the following theorem which can also be found in [3] with a different proof.

Theorem 4.1. *For $n \geq 8$ the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight $n-1$, (and, equivalently, for given n real numbers $x_1, x_2, \ldots, x_n$, the maximum number of subset sums $\sum_{i \in B} x_i$ equal to 0, where no sum of $n-1$ of the given numbers is equal to zero) is 2^{n-2}.*

We will prove the following, somewhat stronger and more precise theorem:

Theorem 4.2. *The maximum number of subset sums $\sum_{i\in B} x_i$ equal to 0, where no sum of $n-1$ of the given numbers is equal to zero is $\max\left(\binom{n}{\lfloor n/2\rfloor}, 2^{n-2}\right)$. More precisely, if we further assume that none of the numbers is equal to zero we have at most $\binom{n}{\lfloor n/2\rfloor}$ 0 subset sums, while if we further assume that the total sum of the numbers is not zero we have at most 2^{n-2} 0 subset sums.*
Equivalently, the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight $n-1$ is $\max\left(\binom{n}{\lfloor n/2\rfloor}, 2^{n-2}\right)$, with being at most $\binom{n}{\lfloor n/2\rfloor}$ if all vertices of weight 1 are avoided further, and at most 2^{n-2} if the vertex $\underline{1} = \{1, 1, \ldots, 1\}$ is avoided as well.

Proof. Note that this later theorem states only — not necessarily sharp — upper bounds. The bounds here can be reached. In case when none of the numbers are 0 — or no vertex of weight 1 is contained in $\mathrm{span}_{\mathbb{R}} M$ — by the original construction of Theorem 2.6 (but this construction will fulfill our main assumption only when n is even, and therefore the bound here might not be sharp) and for the other case by the numbers $1, 1, 0, 0, 0, \ldots, 0$, or, equivalently, choosing all vertices to be in M whose first two coordinates are equal to 0. This later construction always works, so the maximum number we are looking for is always at least 2^{n-2}.

The case when no chosen number can be zero or no vertex of weight 1 can be in $\mathrm{span}_{\mathbb{R}} M$ is the immediate consequence of Theorem 2.6.

The other upper bound (2^{n-2}) will be proven by induction on n, with $n = 3$ being trivial. We will distinguish two cases, if there is a 0 among the chosen numbers (a vertex of weight 1 in $\mathrm{span}_{\mathbb{R}} M$) or if not. In the first case leave that 0 out, resulting $n-1$ numbers, such that the sum of them is not 0 (since it is an $n-1$ subsum of the original numbers) and no subsum of $n-2$ of them is 0 either (since that, together with the left element, would also result an $n-1$ subsum of the original numbers equal to 0). Therefore these $n-1$ numbers will satisfy the assumption and therefore there are at most 2^{n-3} 0 subsums of them. If we add back the left element, being it 0, it can be added to the already 0 subsums, thus doubling the total number of 0-subsums from the original n numbers. The remaining case, that is when there is no 0 among the chosen numbers, will be handled by the following lemma.

Lemma 4.3. *The maximum number of subset sums $\sum_{i\in B} x_i$ equal to 0, where none of the given n numbers is equal to 0, none of the sum of any $n-1$ of them is equal to zero and the total sum of them is neither 0 is 2^{n-2}.*

Note that the lemma itself is a trivial consequence of Theorem 2.6 for most of the values of n, since the condition that none of these numbers are equal to zero already implies the upper limit of $\binom{n}{\lfloor n/2\rfloor}$. However, we need it for small values of n as well (when $\binom{n}{\lfloor n/2\rfloor} > 2^{n-2}$ to have our induction step worked completely.

Proof of the Lemma will be by induction on n again, being trivial for $n = 2, 3$. If chosen numbers have all the same value, that yields no 0 subsum at all, so we may assume there are two different ones of them, x_1 and x_2. Take all the 4 subsets B_1, B_2, B_3 and B_4 of $\{x_1, x_2\}$ and consider the 2^{n-3} pairs of complementary subsets of the remaining $n-2$ numbers. We claim that for any A_1, A_2 of these pairs at most two of the 8 subsets $B_i \cup A_k$ can give 0 sum, therefore the total number of 0 subsums is at most $2 \times 2^{n-3} = 2^{n-2}$.

If $x_1 + x_2 = 0$ then at most one of the A_k's, say A_1 gives a 0 sum (both may not, since than, together with $x_1 + x_2 = 0$ the total sum would be 0), and then A_1 and $A_1 \cup \{x_1, x_2\}$ would give 0 sum. Since the $x_i \neq 0$, the sets $A_1 \cup \{x_i\}$ will not give 0 sum, since A_2 does not give a zero sum, neither does $A_2 \cup \{x_1, x_2\}$ and $A_2 \cup \{x_i\}$ may not give 0 either, since in that case $A_1 \cup A_2 \cup \{x_i\}$ would give a subsum of $n-1$ numbers equal to 0.

If $x_1 + x_2 = 0$ and none of the A_k's give a 0 sum, we can add at most one of x_1 and x_2 to the sum of A_1 to obtain 0, and similarly for A_2, giving again at most 2 0 sums of the form$B_i \cup A_k$.

If $x_1 + x_2 \neq 0$ then the 4 numbers 0, x_1, x_2 and $x_1 + x_2$ are all different, and so at most one of them can be added to A_1 or A_2 to yield a sum equal to 0. ■

Now, Theorem 4.1 is an immediate consequence of Theorem 4.2 observing that for $n \geq 8$ we have $\binom{n}{\lfloor n/2\rfloor} \leq 2^{n-2}$.

The next natural question is to find the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight 2, or, given n real numbers $x_1, x_2, \ldots, x_n$ ($\neq 0$), to find the maximum number of subsums $\sum_{i\in B} x_i = 0$, where the B's are subsets of $\{1, 2, \ldots, n\}$, with none of the sums $x_i + x_j = 0$. This one has a close resemblance to the following well known problem of Erdős and Moser asked after finding the solution of the Littlewood-Offord problem (see Problem 2.11) and it's high symmetry:

Problem 4.4. (Erdős-Moser, [11]) How do we select *distinct* nonzero real numbers $x_1, x_2, \ldots x_n$ and a target sum t to maximize the number of subset sums $= t$?

which is equivalent to

Problem 4.5. How do we select nonzero real numbers $x_1, x_2, \ldots x_n$ to maximize the number of subset sums $= 0$, with all $x_i - x_j \neq 0$?

A possible candidate for the largest such set of numbers is family of the n distinct integers closest to 0 and the target $t = 0$. This was proven to be the best construction by Stanley in [17].

State formally our similar problem in the same language:

Problem 4.6. How do we select (nonzero) real numbers $x_1, x_2, \ldots x_n$ to maximize the number of subset sums $= 0$, with all $x_i + x_j \neq 0$, and what is this maximum?

Note that here the nonzero assumption does not really change the problem, since the condition $x_i + x_j \neq 0$ implies that at most one of the chosen numbers can be equal to zero. If so, remove it and then the remaining $n - 1$ numbers will satisfy the original condition together with the further assumption that none of them are equal to zero. Taking here the best construction, one can always add the last, 0 element to every 0-sum, doubling the number of zero sums. That is, if $m_1(n)$ denotes the maximum number of 0-sums above without assuming that the given numbers are nonzeros and $m_2(n)$ with the additional assumption that they may not be equal to zero, we have that $m_1(n) = 2 \times m_2(n - 1)$.

Both the author, and, independently, Ahlswede, Aydinian, and Khachatrian in [3] conjecture that

Conjecture 4.7. Given n real numbers $x_1, x_2, \ldots x_n$, with all $x_i + x_j \neq 0$ and further, no $x_i = 0$, the maximum number of subset sums $= 0$ ($=m_2(n)$ with the above notation), or, equivalently, the maximum size of M such that $\mathrm{span}_{\mathbb{R}} M$ avoids all vertices of weight 2 and 1 of the hypercube is $\binom{\lceil 2n/3 \rceil}{2} \lfloor n/3 \rfloor$. This bound can be reached by the choice of

$$\{-1, -1, \ldots, -1, 2, \ldots 2\}$$

with $\lceil 2n/3 \rceil$ copies of -1 and $\lfloor n/3 \rfloor$ copies of 2 in the subsum "language" or by choosing M as the set of all vertices of the hypercube which have exactly two 1's among the first $\lceil 2n/3 \rceil$ coordinates and one 1's among the last $\lfloor n/3 \rfloor$ coordinates.

Note that the linear algebra argument already used several times shows here as well that the later choice of M will have the required property: Assume that we are given n numbers $x_1, x_2, \ldots x_n$ and disclose the value of the sum of any three of them, such that two are chosen from the first $\lceil 2n/3 \rceil$ ones and the third from the last $\lfloor n/3 \rfloor$ ones. Increasing the value of the first $\lceil 2n/3 \rceil$ by 1 and decreasing the value of the last $\lfloor n/3 \rfloor$ by 2 will leave the value of all the disclosed subsums unchanged, while the value of any of these numbers and any sum of 2 of these numbers will be changed. Therefore, no linear combination of these triple sums can be equal to any sums of 2 of the numbers.

In the general framework stated in the first paragraph of this chapter, further questions might be asked about the size of M if $\mathrm{span}_{\mathbb{R}} M$ does not contain any vertices of a given constant weight r, or, similarly, about the maximum number of subset sums $\sum_{i \in B} x_i$ equal to 0, for any set of n real numbers $x_1, x_2, \ldots, x_n$, with the condition that none of the sums $\sum_{j=1}^{r} x_{i_j}$ are equal to 0 for a given r. Assuming the validity of Conjecture 4.5 it would not be difficult to prove that for $r = 3$ the best bound is the same as for $r = 1$, that is $\binom{n}{\lfloor n/2 \rfloor}$ with the same construction. Similarly, the $r = 4$ and 5 cases (provided n is big enough compared to k) would be easy to handle with the conjecture, for the even case similar to the $r = 2$ case and the odd to the $r = 1$ case. However, in case of $k = 6$ the construction in the above conjecture does not work, and we have not even a conjecture for the best structure.

Further, asymptotic results might be found in [2] and [3].

A further possible direction of this research to investigate the case when the span of the subset of the vertices is taken over $GF(2)$, for both the unrestricted and restricted cases. Then the subset sum equivalency does not work, we are completely left on linear algebra and other direct tools and many times the answer may depend on the parity of the parameters involved. We have considered only a few of these cases in this paper.

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