KOMPLEX SZÁMOK

FANCY PROBLEMS ABOUT COMPLEX NUMBERS

1. Given a graph (undirected, no loops) and there is a complex number written on each vertex of the graph. Then, the Real Elf snaps. When he snaps, in each vertex the number is replaced by the sum of the numbers on the neighbor vertices. After a while the Real Elf observes that every number z is replaced by λz where λ is fix. Prove that $\lambda \in R$.

2. An n by k chessboard is covered with 1 by r dominos. Prove that either $r|k$ or $r|n$.

3. Evaluate the following sums: $\sum_k \binom{n}{3k}$, $\sum_k (-1)^k \binom{n}{2k}$, $\sum_k \binom{n}{4k}$.

4. Find the sum of the primitive n-th roots of unity.

5. $\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = ?$

6. Let ε be a primitive p-th root of unity, where p is a prime. How much is $|\sum_{k=1}^{p} \varepsilon^{k^2}|$? And $|\sum_{k=1}^{p} \left(\frac{k}{p}\right) \varepsilon^k|$

7. Let $A_1A_2\dots A_n$ be a regular n-gon inscribed in a circle of radius 1. Find $|\overline{A_1A_2}| \cdot |\overline{A_1A_3}| \cdots |\overline{A_1A_n}|$.

8. Prove that the set of natural numbers cannot be partitioned into the union of finitely many disjoint arithmetic progressions with distinct differences.

9. Construct a regular triangle onto each side of an arbitrary triangle. Prove that the centres of these triangles form a regular triangle.

10. Show that the Fermat–conjecture (Wiles' Theorem) is true for complex polynomials, i.e: there are no non-constant coprime polynomials f, g, h such that $f^n + g^n = h^n$.

11. Show that $2(\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} - \sin\frac{\pi}{7}) = \sqrt{7}$ and

$$2(\cos\frac{\pi}{13} + \cos\frac{2\pi}{13} + \cos\frac{3\pi}{13} - \cos\frac{4\pi}{13} - \cos\frac{5\pi}{13} + \cos\frac{6\pi}{13}) = \sqrt{13}.$$